Algebra

Futures Language Schools
Mathematical Department

First Secondary

First Term

Student Name:

Lesson (1)

Solving quadratic equations in one variable

Lesson objectives

Related Links

- Solve quadratic equation in one variable algebraically and graphically.
- Distinguish between equations, relations and functions.

Remarks

① The equation: a x + b = 0 where a $\neq$ 0 is called a **first degree equation in one variable which is** x

② The equation: a x^2 + b x + c = 0 where a $\neq$ 0 is called **a second degree equation in one variable which is** x

General Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The equation: a x^2 + b x + c = 0

where a $\neq$ 0 is of second degree its roots (solutions) gives by the relation

Example ❶

Find in $\mathbb{R}$ the solution set of each of the following equations:

❶ $x^2 - 5x - 6 = 0$

Solution: $\because x^2 - 5x - 6 = 0$ $\therefore (x - 6)(x + 1) = 0$

$\therefore x = 6$ or $x = -1$ $\therefore$ The S.S. = {6 , -1}

❷ $2x^2 + 7x = 0$

Solution: $\because 2x^2 + 7x = 0$ $\therefore x\ (2x + 7) = 0$

$\therefore x = 0$ or $x = \frac{-7}{2}$ $\therefore$ The S.S. = $\{0 , \frac{-7}{2}\}$

❸ $4x^2 - 25 = 0$

Solution: $\because 4x^2 - 25 = 0$ $\therefore (2x-5)(2x+5) = 0$

$\therefore x = \frac{5}{2}$ or $x = \frac{-5}{2}$ $\therefore$ The S.S. = $\{\frac{5}{2}, \frac{-5}{2}\}$

❹ $x^2 - 6x + 9 = 0$

Solution: $\because x^2 - 6x + 9 = 0$ $\therefore (x-3)(x-3) = 0$

$\therefore x = 3$ $\therefore$ The S.S. = $\{3\}$

General Formula:

The equation: a x^2 + b x + c = 0
where a ≠ 0 is of second degree
its roots (solutions) gives by the relation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example ❷

Find in ℝ the solution set of each of the following equations:

❶ $x^2 - 6x - 11 = 0$ (given $\sqrt{5} \simeq 2$)

Solution: $\because x^2 - 6x - 11 = 0$

a = 1
b = – 6
c = −11

$\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = \frac{6 \pm \sqrt{36 - 4 \times 1 \times -11}}{2 \times 1} = \frac{6 \pm \sqrt{80}}{2} = \frac{6 \pm 4\sqrt{5}}{2}$

$\therefore x_1 = \frac{6 + 4\sqrt{5}}{2}$ or $x_2 = \frac{6 - 4\sqrt{5}}{2}$

$\therefore x_1 \simeq \frac{6 + 4 \times 2}{2} = 7$ or $x_2 \simeq \frac{6 - 4 \times 2}{2} = -1$

$\therefore$ The S.S. = $\{7, -1\}$

❷ $x^2 - 6x + 7 = 0$ (given $\sqrt{2} \simeq 1.4$)

Solution: $\because x^2 - 6x + 7 = 0$

a = 1
b = – 6
c = 7

$\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 7}}{2 \times 1} = \frac{6 \pm \sqrt{8}}{2} = \frac{6 \pm 2\sqrt{2}}{2}$

$\therefore x_1 = \frac{6 + 2\sqrt{2}}{2}$ or $x_2 = \frac{6 - 2\sqrt{2}}{2}$

$\therefore x_1 \simeq \frac{6 + 2 \times 1.4}{2} = 2.2$ or $x_2 \simeq \frac{6 - 2 \times 1.4}{2} = 1.6$

$\therefore$ The S.S. = {2.2 , 1.6}

❸ $x - \frac{5}{x} = 3$

Solution: $\because x - \frac{5}{x} = 3$ multiply by x

$\therefore x^2 - 5 = 3x$ $\therefore x^2 - 3x - 5 = 0$

a = 1
b = – 3
c = –5

$\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = \frac{3 \pm \sqrt{9 - 4 \times 1 \times -5}}{2 \times 1} = \frac{3 \pm \sqrt{29}}{2}$

$\therefore x_1 = \frac{3 + \sqrt{29}}{2}$ or $x_2 = \frac{3 - \sqrt{29}}{2}$

$\therefore$ The S.S. = $\left\{\frac{3 + \sqrt{29}}{2}, \frac{3 - \sqrt{29}}{2}\right\}$

❹ $x^2 - 2x + 6 = 0$

Solution: $\because x^2 - 2x + 6 = 0$

a = 1
b = – 2
c = 6

$\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 6}}{2 \times 1} = \frac{2 \pm \sqrt{-20}}{2}$

$\therefore$ The S.S. = $\emptyset$

PRACTICE (1)

Q1: Solve the equation $4x^2 + 40x + 40 = -60$ by factoring.

A $x = -5$

B $x = 1$ or $x = 25$

C $x = -1$ or $x = -25$

D $x = 5$

Q2: Solve the equation $2(x+1)^2 + 5(x+1) = 0$.

A $x = -1, x = -\frac{7}{2}$

B $x = -1, x = -\frac{5}{2}$

C $x = -\sqrt{\frac{5}{2}}$

D $x = 1$

E $x = \sqrt{\frac{5}{2}}$

Q3: Find the solution set of $x(x-19) = -15x$ in $\mathbb{R}$.

A $\{4\}$

B $\{0, 4\}$

C $\{-4\}$

D $\{19, -15\}$

E $\{0, -4\}$

Q4: Find the solution set of $(x+9)^2 = (x+9)$ in $\mathbb{R}$.

A $\{-9, -10\}$

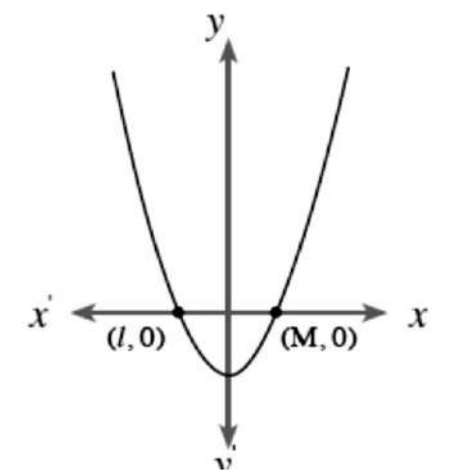
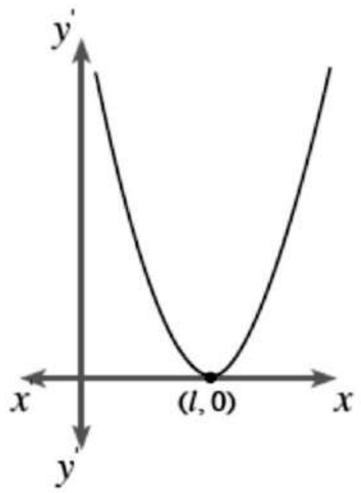
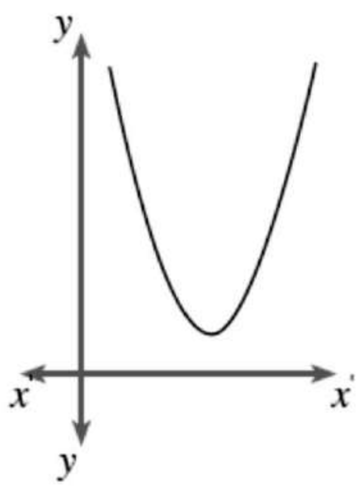
B $\{10\}$

C $\{-9, -8\}$

D $\{9, -8\}$

E $\{9\}$

When solving the quadratic equation in one variable, there are three cases:

① The parabola intersects the x-axis at two points	② The parabola touches the x-axis at one point	③ The parabola does not intersects the x-axis
		
There are two solutions for the equation in $\mathbb{R}$. The solution set = $\{l, \text{M}\}$	*There is a unique solution for the equation in $\mathbb{R}$.* The solution set = $\{ l \}$	*There is no solution for the equation in $\mathbb{R}$.* The solution set = $\emptyset$

Remember that:

The coordinates of the vertex of the quadratic equation: $a x^2 + b x + c = 0$ is:

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2\,a}\right)\right)$$

PRACTICE (2)

Q1: Solve the equation $-x^2 + 7x + 1 = 0$.

A $\left\{\frac{-7+\sqrt{53}}{2}, \frac{-7-\sqrt{53}}{2}\right\}$

B $\left\{\frac{7-\sqrt{53}}{2}, \frac{7+\sqrt{53}}{2}\right\}$

C $\left\{\frac{7-\sqrt{51}}{2}, \frac{7+\sqrt{51}}{2}\right\}$

D $\left\{\frac{49-\sqrt{11}}{2}, \frac{49+\sqrt{11}}{2}\right\}$

E $\left\{\frac{7-\sqrt{11}}{2}, \frac{7+\sqrt{11}}{2}\right\}$

Q2: Find the solution set of the equation $3x^2 - 2(7 - x) = 0$, giving values to one decimal place.

A $\{1.9, -2.5\}$

B $\{-1.9, -2.5\}$

C $\{3.7, -5.0\}$

D $\{-3.7, -5.0\}$

Q3: Find the solution set of the equation $x^2 - 8x - 2 = 9x + 8$, giving values correct to three decimal places.

A $\{17.569, -0.569\}$

B $\{0.569, -17.569\}$

C $\{8.785, -0.285\}$

D $\{0.285, -8.785\}$

Example ❸

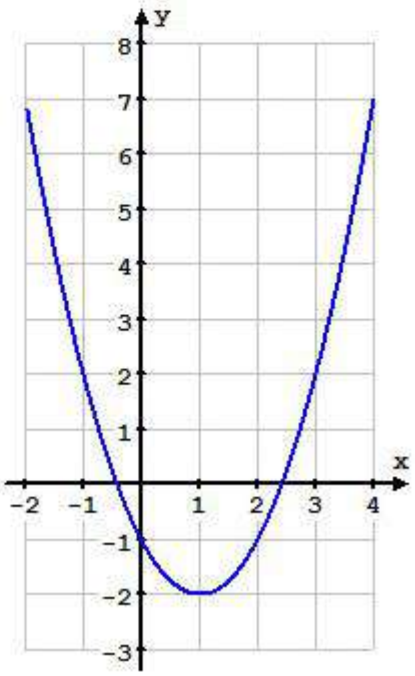
Find the S.S. of each of the following equations graphically , then check the result algebraically:

(1) $x^2 - 2x - 1 = 0$ take $x \in [-2, 4]$

Solution

$f(x) = x^2 - 2x - 1$

x	-2	-1	0	1	2	3	4
$f(x)$	7	2	-1	-2	-1	2	7



$\because x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = \dfrac{2 \pm \sqrt{4 - 4 \times 1 \times -1}}{2 \times 1} = \dfrac{2 \pm \sqrt{8}}{2}$

$\therefore x_1 = \dfrac{2 + 2\sqrt{2}}{2} = 1 + \sqrt{2} \simeq 2.4$

or $x_2 = \dfrac{2 - 2\sqrt{2}}{2} = 1 - \sqrt{2} \simeq -0.4$

$\therefore$ The S.S. = {2.4 , -0.4}

(2) $x^2 + x - 6 = 0$

Solution

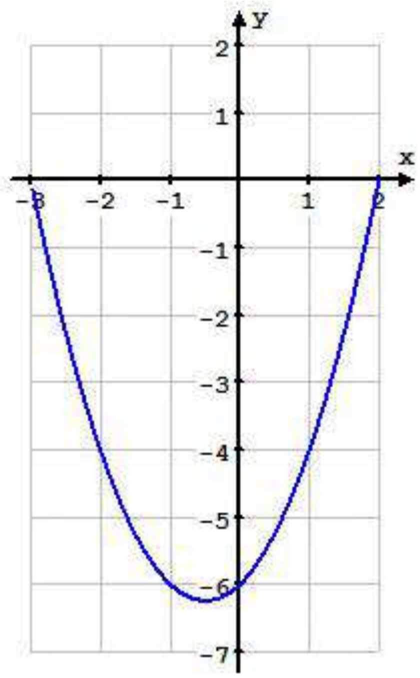
$f(x) = x^2 + x - 6$

The coordinates of the vertex:

$x = \dfrac{-b}{2a} = \dfrac{-1}{2}$

$y = \left(\dfrac{-1}{2}\right)^2 + \left(\dfrac{-1}{2}\right) - 6 = -6\dfrac{1}{4}$

x	-3	-2	-1	$\frac{-1}{2}$	0	1	2
$f(x)$	0	-4	-6	$-6\frac{1}{4}$	-6	-4	0



$\therefore$ The S.S. = {2 , -3}

PRACTICE (3)

Q1: Consider the graph:

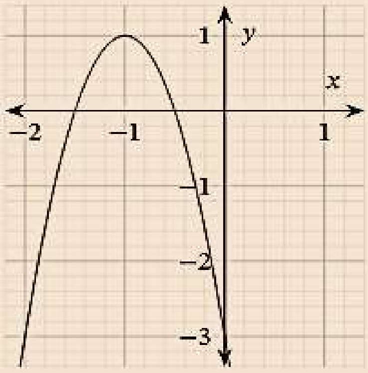
The roots of a quadratic can be read from the graph. What are they?

A 1 and -3

B $-\frac{1}{2}$ and $-\frac{3}{2}$ and -3

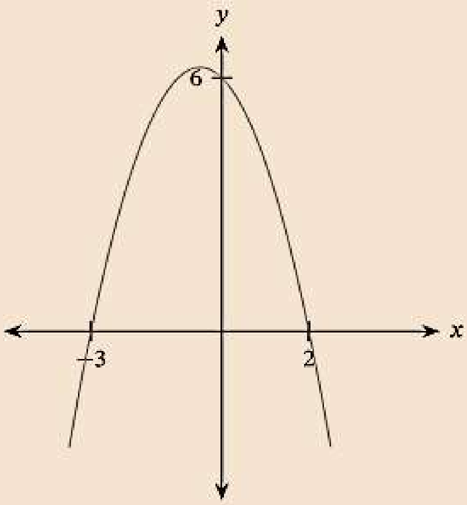
C $-\frac{1}{2}$

D $-\frac{1}{2}$ and $-\frac{3}{2}$

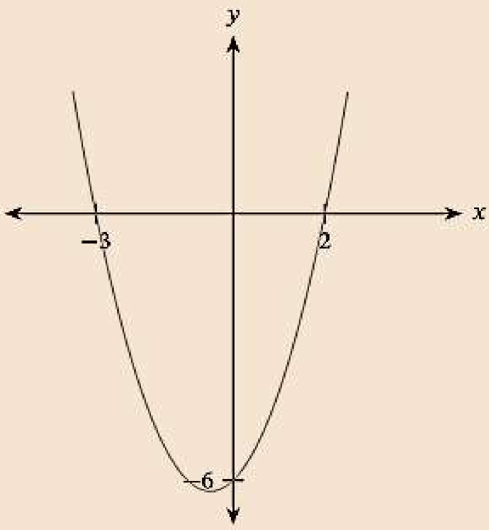


Q2: Solve $-x^2 - x + 6 = 0$ by factoring, and hence determine which of the following figures would be a sketch of $y = -x^2 - x + 6$.

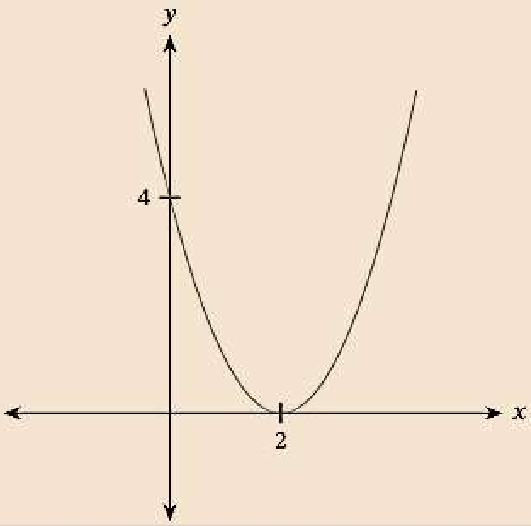
A



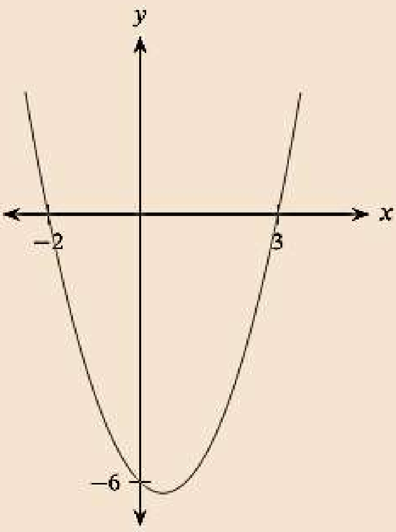
B



C



D



1 - 1 Solving Quadratic Equations in One Variable

First: Multiple choice:

1 The equation: $(x-1)(x+2)=0$ is of: .... degree

A First B second C third D Fourth

2 The solution set of the equation $x^2 = x$ in $\mathbb{R}$ is : ....

A $\{0\}$ B $\{1\}$ C $\{-1, 1\}$ D $\{0, 1\}$

3 The solution set of the equation $x^2 + 3 = 0$ in $\mathbb{R}$ is: ....

A $\{-3\}$ B $\{-\sqrt{3}\}$

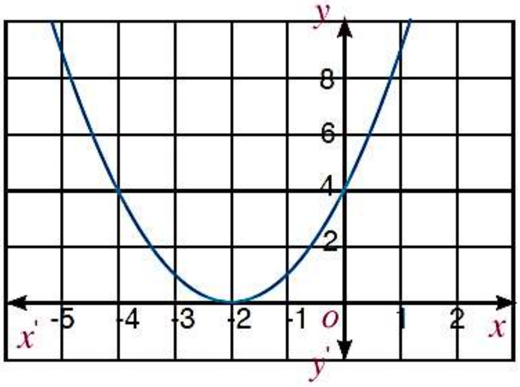
C $\{\sqrt{3}\}$ D ϕ

4 The solution set of the equation $x^2 - 2x = -1$ in $\mathbb{R}$ is: ....

A $\{-1\}$ B ϕ

C $\{-1, 1\}$ D $\{1\}$

5 The figure opposite represents the graph of the curve of the quadratic function f. The solution set of the equation $f(x) = 0$ is: ....



A $\{-2\}$ B $\{4\}$

C ϕ D $\{-2, 4\}$

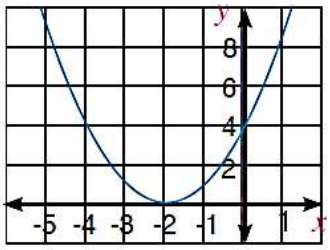
Second: Answer the following questions:

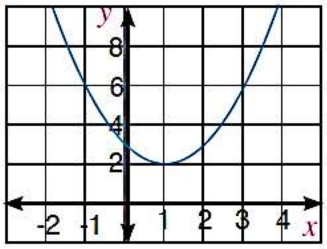
6 Find the solution set of each of the following equations in $\mathbb{R}$:

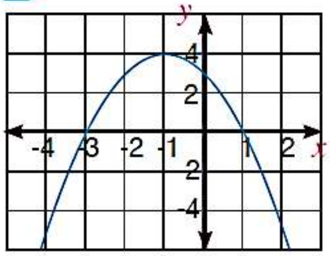
A $x^2 - 1 = 0$ B $x^2 + 3x = 0$ C $(x-4)^2 = 0$

D $x^2 - 6x + 9 = 0$ E $x^2 + 9 = 0$ F $x(x+1)(x-1) = 0$

7) Each of the following graphs illustrates a quadratic function. Find the solution set of the equation f(x) = 0 in each figure .

A


B


C


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8) Find the solution set of each of the following equations in $\mathbb{R}$ then, and verify the result graphically:

A) $x^2 = 3x + 40$ ....

B) $2x^2 = 3 - 5x$ ....

C) $6x^2 = 6 - 5x$ ....

D) $(x - 3)^2 = 5$ ....

E) $x^2 + 2x = 12$ ....

F) $\frac{1}{2}x^2 - \frac{3}{5}x = 1$ ....

9) Solve the following equations in $\mathbb{R}$ using the general formula then approximate the result to the nearest tenth.

A) $3x^2 - 65 = 0$ ....

B) $x^2 - 6x + 7 = 0$ ....

C) $x^2 + 6x + 8 = 0$ ....

D) $2x^2+3x-4 = 0$ ....

E) $5x^2 - 3x - 1 = 0$ ....

F) $3x^2 - 6x - 4 = 0$ ....

(10) **Numbers:** If the sum of the whole consecutive numbers (1+ 2 + 3+ ....+ n)is given by the relation $S = \frac{n}{2}(1+n)$, how many whole consecutive numbers starting from the number 1 and their sum equals:

(A) 78 ....

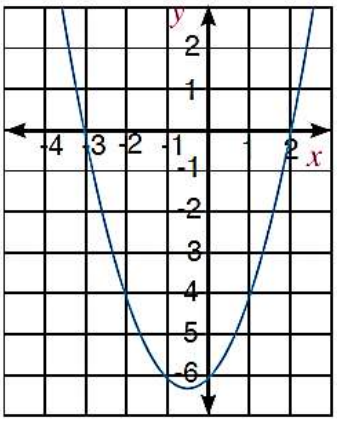
(B) 171 ....

(C) 253 ....

(D) 465 ....

(11) Each of the following figure shows the graph of a quadratic function in one variable. Find the rule of each function.

(A)



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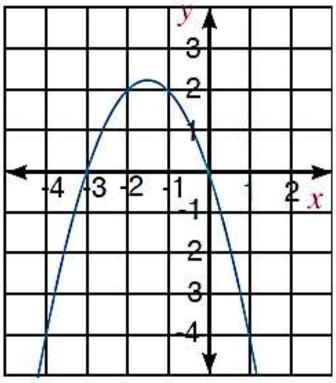
(B)



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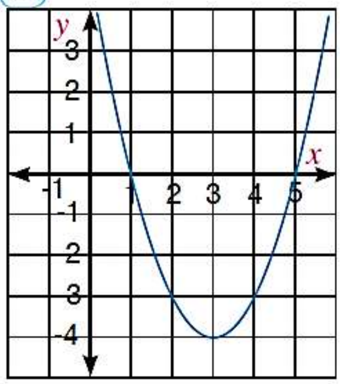
(C)



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Lesson (2) An Introduction in Complex Numbers

Lesson objectives

Related Links

- Identify Equality of two complex numbers.
- Explain Operations on the complex numbers.

Imaginary numbers

The imaginary number "i" is defined as the number whose square equals (-1)
i.e. $i^2 = -1$

For example: $2i$, $-3i$, $\sqrt{3}i$, ………….

Notice that

✦ $\sqrt{-3} = \sqrt{3\,i^2} = \pm\sqrt{3}i$

Write the power on your calculator

÷ 4

Shift S⇔D

$\frac{1}{4} = i^1 = \mathbf{i}$

$\frac{3}{4} = i^3 = -\mathbf{i}$

$\frac{1}{2} = i^2 = -\mathbf{1}$

No fraction = **1**

Integer powers of i

❶ $i^1 = i$	❷ $i^2 = -1$	❸ $i^3 = -i$	❹ $i^4 = 1$
① $i^{4n+1} = i$	② $i^{4n+2} = -1$	③ $i^{4n+3} = -i$	④ $i^{4n} = 1$

Example ❶ <u>**Find each of the following in the simplest form:**</u>

❶ i^{23} ❷ i^{37} ❸ i^{34} ❹ i^{51}

❺ i^{-23} ❻ i^{-35} ❼ i^{-54} ❽ i^{4n+25}

Solution

❶ $i^{23} = i^3 = -i$	❷ $i^{37} = i^1 = i$
❸ $i^{34} = i^2 = -1$	❹ $i^{51} = i^3 = -i$
❺ $i^{-23} = -\,i^{23} = -\,i^3 = -(-i) = i$	❻ $i^{-35} = -\,i^{35} = -\,i^3 = -(-i) = i$
❼ $i^{-54} = -\,i^{54} = -\,i^2 = -(-1) = 1$	❽ $i^{4n+25} = i^{25} = i^1 = i$

Complex number

The complex number is the number which can be written in the form "a + bi" where a and b are real numbers.

Complex number
a + b i
Real part   Imaginary part

Example ❷

Solve each of the following equations:

❶ $5x^2 + 245 = 0$ ❷ $3x^2 + 27 = 0$

❸ $9x^2 + 125 = 61$ ❹ $4x^2 + 100 = 75$

Solution

❶ $5x^2 + 245 = 0$

$\therefore 5x^2 = -245$

$\therefore x^2 = \frac{-245}{5}$

$\therefore x^2 = -49$ $\therefore x = \pm\sqrt{-49}$

$\therefore x = \pm\sqrt{49\ \mathrm{i}^2}$ $\therefore x = \pm 7\ \mathrm{i}$

❷ $3x^2 + 27 = 0$

$\therefore 3x^2 = -27$

$\therefore x^2 = \frac{-27}{3}$

$\therefore x^2 = -9$ $\therefore x = \pm\sqrt{-9}$

$\therefore x = \pm\sqrt{9\ \mathrm{i}^2}$ $\therefore x = \pm 3\ \mathrm{i}$

❸ $9x^2 + 125 = 61$

$\therefore 9x^2 = 61 - 125$

$\therefore 9x^2 = -64$

$\therefore x^2 = \frac{-64}{9}$ $\therefore x = \pm\sqrt{\frac{-64}{9}}$

$\therefore x = \pm\sqrt{\frac{64}{9}\ \mathrm{i}^2}$ $\therefore x = \pm\frac{8}{3}\mathrm{i}$

❹ $4x^2 + 100 = 75$

$\therefore 4x^2 = 75 - 100$

$\therefore 4x^2 = -25$

$\therefore x^2 = \frac{-25}{4}$ $\therefore x = \pm\sqrt{\frac{-25}{4}}$

$\therefore x = \pm\sqrt{\frac{25}{4}\ \mathrm{i}^2}$ $\therefore x = \pm\frac{5}{2}\mathrm{i}$

PRACTICE (1)

Q1: Solve the equation $2x^2 = -50$.

A $x = 5, x = -5$

B $x = 5i, x = -5i$

C $x = 5\sqrt{2}i, x = -5\sqrt{2}i$

D $x = \frac{5\sqrt{2}}{2}i, x = -\frac{5\sqrt{2}}{2}i$

E $x = \frac{5}{2}i, x = -\frac{5}{2}i$

Q2: Simplify $17i(-5i)$.

Q3: Simplify $(2i)^2(-2i)^3$.

A 32

B $-32i$

C 4

D $-4i$

E $32i$

Q4: Solve the equation $x^2 = -16$.

A $x = -8$

B $x = 2i, x = -2i$

C $x = 4i, x = -4i$

D $x = i, x = 4$

E $x = 4, x = -4$

Equality of two complex numbers:

Two complex numbers are equal if and only if the two real parts are equal and the two imaginary parts are equal.

If: a + b i = c + d i **then**: $\boxed{a = c}$ and $\boxed{b = d}$ and ***vice versa***.

Example ❸

Find the values of *x* and *y* which satisfy each of the following equations:

❶ $(2x + 1) + 4y\,i = 5 - 12\,i$

❷ $2x - y + (x - 2y)\,i = 5 + i$

❸ $2x - 3 + (3y + 1)\,i = 7 + 10\,i$

❹ $3x\,i + 5 - 3\,i - 2\,y = 2 - i$

Solution

❶ $\because (2x + 1) + 4y\,i = 5 - 12\,i$ $\therefore (2x + 1) = 5$ & $4y = -12$ $\therefore 2x = 5 - 1 = 4$ $\quad y = \frac{-12}{4}$ $\therefore x = 2$ $\quad y = -3$	❷ $\because 2x - y + (x - 2y)\,i = 5 + i$ $\therefore (2x - y) = 5$ & $(x - 2y) = 1$ By using calculator: $\boxed{\text{Mode}} \to \boxed{\text{EQN}} \to \boxed{1}$ $\therefore x = 3$ $\quad y = 1$
❸ $\because 2x - 3 + (3y + 1)\,i = 7 + 10\,i$ $\therefore (2x - 3) = 7$ & $(3y + 1) = 10$ $\therefore 2x = 7 + 3 = 10$ $\quad 3y = 10 - 1 = 9$ $\therefore x = 5$ $\quad y = 3$	❹ $\because 3x\,i + 5 - 3\,i - 2\,y = 2 - i$ $\therefore (3x - 3) = -1$ & $(5 - 2y) = 2$ $\therefore 3x = -1 + 3 = 2$ $\quad -2y = 2 - 5 = -3$ $\therefore x = \frac{2}{3}$ $\quad y = \frac{3}{2}$

Operations on complex numbers

Example 4

Find in the simplest form the result of each of the following:

1. $(7 - 4i) + (2 + i)$
2. $(12 - 5i) - (7 - 9i)$
3. $(2 + 3i)(3 - 4i)$
4. $(4 - 3i)(4 + 3i)$
5. $(5 - 6i)(3 + 2i)$
6. $(-2i + 5)(3 + 2i)$
7. $(4 - 3i)(5 - 2i)$
8. $(3i - 4)(2 + 3i)$

Solution

1. $(7 - 4i) + (2 + i) = 9 - 2i$	2. $(12 - 5i) - (7 - 9i) = 5 + 4i$
3. $(2 + 3i)(3 - 4i) = 6 - 8i + 9i + 12 = 18 + i$	
4. $(4 - 3i)(4 + 3i) = 16 + 9 = 25$	
5. $(5 - 6i)(3 + 2i) = 15 + 10i - 18i + 12 = 27 - 8i$	
6. $(-2i + 5)(3 + 2i) = -6i + 4 + 15 + 10i = 19 + 4i$	
7. $(4 - 3i)(5 - 2i) = 20 - 8i - 15i - 6 = 14 - 23i$	
8. $(3i - 4)(2 + 3i) = 6i - 9 - 8 - 12i = -17 - 6i$	

PRACTICE (2)

Q1: Simplify $14-(9-8i)+(3-12i)-(9-4i)$.

A $35-24i$

B -1

C $21-24i$

D $3-16i$

Q2: If the complex numbers $7+ai$ and $b-3i$ are equal, what are the values of a and b?

A $a=3, b=-7$

B $a=-3, b=-7$

C $a=7, b=-3$

D $a=-7, b=3$

E $a=-3, b=7$

Q3: What is $(-7-i)-(3-4i)+(2-7i)$?

A $-2-12i$

B $-6+2i$

C $-12+10i$

D $-8-4i$

Conjugate Numbers:

The two numbers a + b i and a – b i are called conjugate numbers.

For example: 4 – 3i and 4 + 3i are two conjugate numbers.

Example ❺

Find in the simplest form, the value of each of the following:

❶ $\frac{4-6i}{2i}$ ❷ $\frac{26}{3-2i}$ ❸ $\frac{3-i}{2-i}$ ❹ $\frac{3+4i}{5-2i}$

Solution

❶ $\frac{4-6i}{2i} = \frac{(4-6i)}{2i} \times \frac{i}{i} = \frac{4i+6}{-2} = -3-2i$

❷ $\frac{26}{3-2i} = \frac{26}{(3-2i)} \times \frac{(3+2i)}{(3+2i)} = \frac{78+52i}{9+4} = \frac{78+52i}{13} = 3+2i$

❸ $\frac{3-i}{2-i} = \frac{(3-i)}{(2-i)} \times \frac{(2+i)}{(2+i)} = \frac{6+3i-2i+1}{4+1} = \frac{7+i}{5} = \frac{7}{5} + \frac{1}{5}i$

❹ $\frac{3+4i}{5-2i} = \frac{(3+4i)}{(5-2i)} \times \frac{(5+2i)}{(5+2i)} = \frac{15+6i+20i-8}{25+4} = \frac{7+26i}{29} = \frac{7}{29} + \frac{26}{29}i$

Example ❻

Find the values of x and y which satisfy the equation:

$$\frac{(2+i)(2-i)}{3+4i} = x + iy$$

Solution

$$\frac{(2+i)(2-i)}{3+4i} = \frac{4+1}{3+4i} = \frac{5}{3+4i} = \frac{5}{(3+4i)} \times \frac{(3-4i)}{(3-4i)} = \frac{15-20i}{9+16} = \frac{15-20i}{25} = \frac{3}{5} - \frac{4}{5}i$$

$\therefore x = \frac{3}{5}$ $\therefore y = -\frac{4}{5}$

PRACTICE (3)

Q1: Expand and simplify $(4 - i)(3 + 2i)$.

A $14 + 5i$

B $12 + 3i$

C $12 - 2i$

D $12 + 7i$

E $10 + 5i$

Q2: Multiply $(-3 + i)$ by $(2 + 5i)$.

A $-6 + 5i$

B $-6 - 18i$

C $-1 - 13i$

D $-6 - 8i$

E $-11 - 13i$

Q3: Simplify $(3 - 6i)^2(2 - i)$.

A $-15i$

B $-25 - 37i$

C $8 - 11i$

D $-90 - 45i$

E $-27 - 36i$

PRACTICE (4)

Q1: Simplify $\frac{2}{3+i}$.

A $\frac{(3-i)}{10}$

B $\frac{(3-i)}{5}$

C $3-i$

D $\frac{2}{3}+2i$

E $\frac{2}{3}-2i$

Q2: Put $\frac{-18-9i}{3i}$ in the form $a+bi$.

A $-3+6i$

B $27-54i$

C $3+54i$

D $-9+18i$

Q3: Simplify $\frac{3-6i}{1-5i}$.

A $\frac{33}{26}+\frac{9}{26}i$

B $-\frac{9}{13}-\frac{5}{26}i$

C $-\frac{11}{8}-\frac{3}{8}i$

D $\frac{3}{4}+\frac{5}{24}i$

E $3+\frac{6}{5}i$

Complex Numbers

1 - 2

1 Simplify:

A i^{66} ....

B i^{-45} ....

C i^{4n+2} ....

D i^{4n-1} ....

2 Simplify:

A $\sqrt{-18} \times \sqrt{-12}$ ....

B $3\,i\,(-2i)$ ....

C $(-4\,i)(-6\,i)$ ....

D $(-2\,i)^3\,(-3\,i)^2$

3 Find in the simplest form:

A $(3 + 2i) + (2 - 5\,i)$ ....

B $(26 - 4i) - (9 - 20\,i)$ ....

C $(20 + 25\,i) - (9 - 20\,i)$ ....

4 Rewrite each of the following in the form $a + b\,i$

A $(2 + 3\,i) - (1 - 2i)$

B $(1 + 2i^3)\,(2 + 3\,i^5 + 4\,i^6)$

5 Rewrite each of the following in the form $a + b\,i$

A $\dfrac{2}{1+i}$ ....

B $\dfrac{4+i}{i}$ ....

C $\dfrac{2-3i}{3+i}$ ....

D $\dfrac{(3+i)(3-i)}{3-4\,i}$ ....

6 Solve each of the following equations:

A $3x^2 + 12 = 0$ ....

B $4y^2 + 20 = 0$ ....

C $4z^2 + 72 = 0$ ....

D $\frac{3}{5}\,y^2 + 15 = 0$ ....

7 **Electricity:** find the total current intensity of the electric current passing through two resistances connected in parallel in a closed circuit if the current intensity in the first resistance is $(4 - 2i)$ ampere and in the second one is $\dfrac{6+3i}{2+i}$ ampere ....

Lesson (3)

Determining the types of roots of a quadratic equation

Lesson objectives

Related Links

- determine the type of the two roots of the quadratic equation

Discriminant $\Delta = b^2 - 4ac$

+ ve	- ve	Zero
The equation has two different real roots	**The equation has two complex roots**	**The equation has two equal real roots**

Example ❶ **Investigate the kind of the roots of the equation.**

① $x^2 - 2x + 5 = 0$ ② $x^2 - 10x + 25 = 0$

③ $3x^2 + 10x - 4 = 0$ ④ $x^2 + 2x + 2 = 0$

Solution

① $x^2 - 2x + 5 = 0$

a = 1	b = −2	c = 5

$\because \Delta = b^2 - 4ac = (-2)^2 - 4 \times 1 \times 5 = -16$

$\therefore$ The equation has two complex roots

② $x^2 - 10x + 25 = 0$

a = 1	b = −10	c = 25

$\because \Delta = b^2 - 4ac = (-10)^2 - 4 \times 1 \times 25 = 0$

$\therefore$ The equation has two equal real roots

③ $3x^2 + 10x - 4 = 0$

a = 3	b = 10	c = −4

$\because \Delta = b^2 - 4ac = (10)^2 - 4 \times 3 \times -4 = 148$

$\therefore$ The equation has two different real roots

④ $x^2 + 2x + 2 = 0$

a = 1	b = 2	c = 2

$\because \Delta = b^2 - 4ac = (2)^2 - 4 \times 1 \times 2 = -4$

$\therefore$ The equation has two complex roots

Example ❷

If the two roots of the equation: $x^2 - kx + 2k - 4x + 5 = 0$ are equal.

Find the real value of k and hence find the two roots.

Solution

$x^2 - kx + 2k - 4x + 5 = 0$

a = 1	b = −k −4	c = 2 k + 5

$\because$ The two roots of the equation are equal $\quad \therefore \Delta = 0$

$\because \Delta = b^2 - 4ac = (-k-4)^2 - 4 \times 1 \times (2k+5) = 0$

$\therefore k^2 + 8k + 16 - 8k - 20 = 0 \quad \therefore k^2 - 4 = 0$

$\therefore k^2 = 4 \quad \therefore k = \pm\sqrt{4} = \pm 2$

If k = 2	If k = −2
$\because x^2 - kx + 2k - 4x + 5 = 0$	$\because x^2 - kx + 2k - 4x + 5 = 0$
$\therefore x^2 - (2)x + 2(2) - 4x + 5 = 0$	$\therefore x^2 - (-2)x + 2(-2) - 4x + 5 = 0$
$\therefore x^2 - 6x + 9 = 0$	$\therefore x^2 - 2x + 1 = 0$
$\therefore x = 3$	$\therefore x = 1$

$a x^2 + b x + c = 0$

Let the two roots L and M:

Sum of roots $= \frac{-b}{a} = \frac{-\text{coeff of } x}{\text{coeff of } x^2}$	Product of roots $= \frac{c}{a} = \frac{\text{Free Term}}{\text{coeff of } x^2}$

Notes:

1. Additive inverse.	$(L, -L)$
2. Multiplicative inverse.	$(L, \frac{1}{L})$
3. Double (twice).	$(L, 2L)$
4. Three times the other.	$(L, 3L)$
5. Exceeds the other by 3.	$(L, L+3)$

Example ❸

<u>Find the sum and product of two roots:</u>

① $2x^2 + 3x - 5 = 0$

Sum of roots $= \frac{-b}{a} = \frac{-3}{2}$	**Product of roots** $= \frac{c}{a} = \frac{-5}{2}$

② $3x^2 + 5 = 4x$

$3x^2 - 4x + 5 = 0$

Sum of roots $= \frac{-b}{a} = \frac{4}{3}$	**Product of roots** $= \frac{c}{a} = \frac{5}{3}$

③ $3x^2 - 12x - 17 = 0$

Sum of roots $= \frac{-b}{a} = \frac{12}{3} = 4$	**Product of roots** $= \frac{c}{a} = \frac{-17}{3}$

Example ❹

Find the value of β which make one of the two roots of the equation: $x^2 + \beta x - 50 = 0$ twice the additive inverse of the other root.

Solution

Let the two roots be: L and -2L

$\because$ **Sum of roots** $= \frac{-b}{a} = \frac{-\beta}{1} = -\beta$ $\quad$ $\because$ **Sum of roots (L & -2L)** $= -\text{L}$

$\therefore -\beta = -\text{L}$ $\quad$ $\boxed{\therefore \beta = \text{L}}$

$\because$ **Product of roots** $= \frac{c}{a} = \frac{-50}{1} = -50$ $\quad$ $\because$ **Product of roots (L & -2L)** $= -2\text{L}^2$

$\therefore -50 = -2\text{L}^2$ $\quad$ $\therefore \text{L}^2 = 25$

$\therefore \beta^2 = 25$ $\quad$ $\therefore \beta = \pm\sqrt{25} = \pm 5$

Example ❺

If $(a-2)\,x^2+(a+1)\,x=6$, then find the value of a in each of the following:

i) The sum of its roots is 3.

ii) The product of the two roots is -6.

Solution

$\because (a-2)\,x^2+(a+1)\,x=6$ $\therefore (a-2)\,x^2+(a+1)\,x-6=0$

i) The sum of its roots is 3.

$\because$ Sum of roots $=\frac{-b}{a}=\frac{-(a+1)}{(a-2)}=3$

$\therefore 3\,(a-2)=-(a+1)$ $\therefore 3a-6=-a-1$

$\therefore 3a+a=-1+6$ $\therefore 4a=5$ $\therefore a=\frac{5}{4}$

ii) The product of the two roots is -6.

$\because$ Product of roots $=\frac{c}{a}=\frac{-6}{(a-2)}=-6$

$\therefore (a-2)=1$ $\therefore a=1+2$ $\therefore a=3$

Example ❻

Find the satisfying condition which makes one of the roots of the equation: $a\,x^2+b\,x+c=0$ equal the additive inverse of twice the other root.

Solution

Let the two roots be: L and $-2L$

$\because$ Sum of roots $=\frac{-b}{a}=-L$ $\boxed{\therefore L=\frac{b}{a}}$

$\because$ Product of roots $=\frac{c}{a}=-2L^2$

$\therefore \frac{c}{a}=-2\left(\frac{b}{a}\right)^2$ $\therefore \frac{c}{a}=-\frac{2\,b^2}{a^2}$ $\therefore c=-\frac{2\,b^2}{a}$

$\therefore a\,c=-2\,b^2$ $\boxed{\therefore a\,c+2\,b^2=0}$ *The require condition*

PRACTICE (1)

Q1: Given that m is a real number, and the equation $(4m+8)x^2-4mx+m=0$ does not have real roots, find the interval which contains m.

A $]-\infty,0]$

B $]-\infty,32]$

C $]-\infty,0[$

D $[0,\infty[$

E $]0,\infty[$

Q2: If the roots of the equation $x^2-8(k+1)x+64=0$ are equal, find the possible values of k.

A $\{-3,1\}$

B $\{-1\}$

C $\{3,-1\}$

D $\{1,-1\}$

E $\{-33\}$

Q3: If the roots of the equation $4x^2-kx+1=0$ are equal, what are the possible values of k?

A $4,-4$

B -4

C $12,-12$

D 12

Q4: Given that the equation $x^2-(-2m+28)x+m^2=0$ has no real roots, find the interval that contains m.

A $m\in[7,\infty[$

B $m\in]-\infty,7]$

C $m\in]7,\infty[$

D $m\in]-\infty,7[$

1 - 3 Determining the type of roots of a Quadratic equation

First: Multiple choice:

(1) The two roots of the equation $x^2 - 4x + k = 0$ are equal if: ....

(A) $k = 1$ (B) $k = 4$ (C) $k = 8$ (D) $k = 16$

(2) The two roots of the equation $x^2 - 2x + M = 0$ are real different if : ....

(A) $M = 1$ (B) $M < 1$ (C) $M > 1$ (D) $M = 4$

(3) The two roots of the equation $L\,x^2 - 12x + 9 = 0$ are complex and not real if : ....

(A) $L > 4$ (B) $L < 4$ (C) $L = 4$ (D) $L = 1$

Second: Answer the following questions:

(4) Determine the number of roots and their types in the following quadratic equation:

(A) $x^2 - 2x + 5 = 0$ ....
(B) $3x^2 + 10x - 4 = 0$ ....
(C) $x^2 - 10x + 25 = 0$ ....
(D) $6x^2 - 19x + 35 = 0$ ....
(E) $(x - 11) - x(x - 6) = 0$ ....
(F) $(x - 1)(x - 7) = 2(x - 3)(x - 4)$ ....

(5) Find the solution of the following equations in the set of complex numbers using the general formula.

(A) $x^2 - 4x + 5 = 0$ ....
(B) $2x^2 + 6x + 5 = 0$ ....
(C) $3x^2 - 7x + 6 = 0$ ....
(D) $4x^2 - x + 1 = 0$ ....

6) Find the value of K in each of the following cases:

A) If the two roots of the equation $x^2 + 4x + K = 0$ are real different.

....

B) If the two roots of the equation $x^2 - 3x + 2 + \frac{1}{K} = 0$ are equal.

....

C) If the two roots of the equation $K x^2 - 8x + 16 = 0$ are complex and not real.

....

7) If L and M are two rational numbers, then prove that the two roots of the equation:
$L x^2 + (L - M) x - M = 0$ are two rational numbers .

....

8) Population of Egypt in 2013 is estimated by the relation:
$Z = n^2 + 1.2\ n + 91$ where (n) is the number of years and (z) is the number of populations in millions.

A) What is the population in 2013? ....

B) Estimate the population in 2023. ....

C) Estimate the number of years at which the population will be 334 million. ....

D) Write a report showing the reasons for which the population is increasing and the way of its treatment.

9) **Discover the error:** What is the number of solutions of the equation $2x^2 - 6x = 5$ in R

Ahmed's answer

$b^2 - 4ac = (-6)^2 - 4 \times 2 \times 5$

$= 36 - 40 = -4$

The discriminant is negative, then there is no real solutions

Karim's answer

$b^2 - 4ac = (-6)^2 - 4 \times 2\ (-5)$

$= 36 + 40 = 76$

the discriminant is positive, then there are two real different solutions

....

....

10) If the two roots of the equation $x^2 + 2\ (K - 1)\ x + (2K + 1) = 0$ are equal, then find the real values of K , and the two roots.

Lesson (4)

Relation between the Two Roots of the Second Degree Equation and the Coefficients of its Terms

Lesson objectives

Related Links

- Find the sum of the two roots of a given quadratic equation.
- Find the product of the two roots.
- Find a quadratic equation in terms of another quadratic equation

$$x^2 - (\text{sum of roots})\, x + \text{product roots} = 0$$

Example ❶

Form the quadratic equation whose roots are:

(i) $2, -3$ **(ii)** $6i, -6i$ **(iii)** $\frac{-2+2i}{1+i}, \frac{-2-4i}{2-i}$

Solution

(i) $2, -3$

$\because$ Sum of roots $= -1$

$\because$ Product of roots $= -6$

$\therefore$ **The equation:** $x^2 + x - 6 = 0$

(ii) $6i, -6i$

$\because$ Sum of roots $= 0$

$\because$ Product of roots $= 36$

$\therefore$ **The equation:** $x^2 + 36 = 0$

(iii) $\frac{-2+2i}{1+i}, \frac{-2-4i}{2-i}$

$$\frac{-2+2i}{1+i} = \frac{(-2+2i)}{(1+i)} \times \frac{(1-i)}{(1-i)} = \frac{-2+2i+2i+2}{1+1} = \frac{4i}{2} = 2i$$

$$\frac{-2-4i}{2-i} = \frac{(-2-4i)}{(2-i)} \times \frac{(2+i)}{(2+i)} = \frac{-4-2i-8i+4}{4+1} = \frac{-10i}{5} = -2i$$

$\because$ Sum of roots $= 0$

$\because$ Product of roots $= 4$

$\therefore$ **The equation:** $x^2 + 4 = 0$

PRACTICE (1)

Q1: Find, in its simplest form, the quadratic equation whose roots are $8\sqrt{11}$ and $-\sqrt{11}$.

A $x^2 - 7\sqrt{11}x - 88 = 0$

B $x^2 + 7\sqrt{11}x - 88 = 0$

C $x^2 - 7x - 88 = 0$

D $x^2 - 7\sqrt{11}x + 88 = 0$

E $-88x^2 - 7\sqrt{11}x - 88 = 0$

Q2: Find, in its simplest form, the quadratic equation whose roots are $m + 3n$ and $m - 3n$.

A $x^2 - 2mx + m^2 - 9n^2 = 0$

B $x^2 + 2x + m^2 - 9n^2 = 0$

C $x^2 - 6nx + m^2 - 9n^2 = 0$

D $x^2 - 2x + m^2 + 9n^2 = 0$

E $x^2 - 2mx + m^2 - 3n^2 = 0$

Q3: What is the simplest form of the quadratic equation whose roots are $\frac{13}{2}$ and $\frac{5}{3}$?

A $6x^2 + 29x - 65 = 0$

B $6x^2 + 49x + 65 = 0$

C $2x^2 - 3x - 65 = 0$

D $6x^2 - 49x + 65 = 0$

Example ❷

If α, β are the roots of the equation $x^2 - 6x + 10 = 0$
Form the equation whose roots are:

(a) $\alpha + 2, \beta + 2$ **(b)** α^2, β^2

(c) $\frac{1}{\alpha}, \frac{1}{\beta}$ **(d)** $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Procedure

① $\alpha + \beta = \frac{-b}{a} = 6$

② $\alpha\beta = \frac{c}{a} = 10$

$\alpha^2 + \beta^2 = 6^2 - 2(10)$

③ $\alpha^2 + \beta^2 = 16$

Solution

(a) $\alpha + 2, \beta + 2$

$\because$ The sum of roots $= (\alpha + 2) + (\beta + 2) = \alpha + \beta + 4 = 6 + 4 = 10$

$\because$ The product of roots $= (\alpha + 2)(\beta + 2) = \alpha\beta + 2\alpha + 2\beta + 4$

$= \alpha\beta + 2(\alpha + \beta) + 4 = 10 + 2 \times 6 + 4 = 26$

$\therefore$ **The equation is:** $\boxed{x^2 - 10x + 26 = 0}$

(b) α^2, β^2

$\because$ The sum of roots $= \alpha^2 + \beta^2 = 16$

$\because$ The product of roots $= \alpha^2 \times \beta^2 = (\alpha\beta)^2 = 10^2 = 100$

$\therefore$ **The equation is:** $\boxed{x^2 - 16x + 100 = 0}$

(c) $\frac{1}{\alpha}, \frac{1}{\beta}$

$\because$ The sum of roots $= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{6}{10} = \frac{3}{5}$

$\because$ The product of roots $= \frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{10}$

$\therefore$ **The equation is:** $x^2 - \frac{3}{5}x + \frac{1}{10} = 0 \quad \times 10$ $\boxed{10x^2 - 6x + 1 = 0}$

(d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

$\because$ The sum of roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{16}{10} = \frac{8}{5}$

$\because$ The product of roots $= \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$

$\therefore$ **The equation is:** $x^2 - \frac{8}{5}x + 1 = 0 \quad \times 5$ $\boxed{5x^2 - 8x + 5 = 0}$

Example ❸

If L & M are the roots of the equation a x^2 + b x + b = 0

Prove that $\frac{1}{L}+\frac{1}{M}+1=0$ and form the equation whose roots are $\frac{1}{L}$, $\frac{1}{M}$.

Solution

$\because L+M=\frac{-b}{a}$ $\qquad \because L\,M=\frac{b}{a}$

$\therefore \frac{1}{L}+\frac{1}{M}=\frac{M+L}{L\,M}=(L+M)\div(L\,M)=\frac{-b}{a}\div\frac{b}{a}=-1$ $\qquad \therefore \frac{1}{L}+\frac{1}{M}+1=0$

$\because$ The sum of roots $=\frac{1}{L}+\frac{1}{M}=\frac{M+L}{L\,M}=-1$

$\because$ The product of roots $=\frac{1}{L}\times\frac{1}{M}=\frac{1}{L\,M}=\frac{a}{b}$

$\therefore$ The equation is: $x^2+x+\frac{a}{b}=0$ $\qquad \times\, b$

$\therefore$ **The equation is:** $\boxed{b\,x^2+b\,x+a=0}$

Example ❹

If the ratio between the two roots of the equation:

x^2 + a x + b = 0 equals 2 : 3 **prove that**: 25 b = 6 a^2

Solution

Let the two roots be: 2 α & 3 α

$\because$ **The sum of roots** $=\frac{-a}{1}=-a$ $\qquad \because$ **Sum of roots** $=2\alpha+3\alpha=5\alpha$

$\therefore \boxed{5\alpha=-a}$ $\qquad \therefore \boxed{\alpha=-\frac{a}{5}}$

$\because$ **The product of roots** $=\frac{b}{1}=b$ $\qquad \because$ **Product of roots** $=2\alpha\times3\alpha=6\alpha^2$

$\therefore \boxed{6\alpha^2=b}$ $\qquad \therefore 6\left(-\frac{a}{5}\right)^2=b$

$\therefore 6\times\frac{a^2}{25}=b$ $\qquad \therefore 25\,b=6\,a^2$

Example ❺

If L, M are the roots of equation:

$3 - 2x - x^2 = 0$ Form the equation whose

Roots are $L + \frac{1}{M}$, $M + \frac{1}{L}$

Procedure

① $L + M = \frac{-b}{a} = \frac{-2}{1} = -2$

② $L\,M = \frac{c}{a} = \frac{-3}{1} = -3$

$L^2 + M^2 = (-2)^2 - 2(-3)$

③ $L^2 + M^2 = 10$

Solution

$3 - 2x - x^2 = 0 \quad \Rightarrow \quad x^2 + 2x - 3 = 0$

$\because$ The sum of roots $= \left(L + \frac{1}{M}\right) + \left(M + \frac{1}{L}\right) = (L + M) + \left(\frac{1}{M} + \frac{1}{L}\right)$

$= (L + M) + \left(\frac{L + M}{L\,M}\right) = -2 + \left(\frac{-2}{-3}\right) = \frac{4}{3}$

$\because$ The product of roots $= \left(L + \frac{1}{M}\right)\left(M + \frac{1}{L}\right) = L\,M + 1 + 1 + \frac{1}{L\,M}$

$= -3 + 1 + 1 + \frac{1}{-3} = -\frac{4}{3}$

$\therefore$ The equation is: $x^2 - \frac{4}{3}x - \frac{4}{3} = 0 \qquad \times 3$

$\therefore$ **The equation is:** $\boxed{3x^2 - 4x - 4 = 0}$

Example ❻

If (1 + i) is one of the roots of the equation $x^2 - 2x + a = 0$ where a ∈ R*
then find: (A) The other root (B) the value of a

Solution

Let the other root be: L

$\because$ Sum of roots $= \frac{2}{1} = 2$ $\qquad \therefore (1 + i) + L = 2$

$\therefore L = 2 - (1 + i) = (1 - i)$ $\qquad \therefore$ The other root $= (1 - i)$ $\rightarrow$ **(A)**

$\because$ Product of roots $= \frac{a}{1} = a$ $\qquad \therefore (1 + i)(1 - i) = a$

$\therefore 1 + 1 = a$ $\qquad \therefore a = 2$ $\rightarrow$ **(B)**

PRACTICE (2)

Q1: Given that $L+3$ and $M+3$ are the roots of the equation $x^2+8x+12=0$, find, in its simplest form, the quadratic equation whose roots are L and M.

A $x^2+17x+34=0$

B $x^2+14x+45=0$

C $x^2-17x+31=0$

D $x^2-17x+19=0$

E $x^2+17x+19=0$

Q2: Given that L and M are the roots of the equation $x^2-2x+5=0$, find, in its simplest form, the quadratic equation whose roots are L^2 and M^2.

A $x^2+14x+25=0$

B $x^2+8x+25=0$

C $x^2-6x+25=0$

D $x^2+6x+25=0$

E $x^2-6x+10=0$

Q3: Given that L and M are the roots of the equation $x^2-3x+12=0$, find, in its simplest form, the quadratic equation whose roots are $\frac{1}{L^2}$ and $\frac{1}{M^2}$.

A $x^2-15x+1=0$

B $144x^2-15x+1=0$

C $144x^2+15x+1=0$

D $144x^2+5x+1=0$

E $144x^2-15x-1=0$

Q4: Given that L and M are the roots of the equation $x^2 - 13x - 5 = 0$, find, in its simplest form, the quadratic equation whose roots are $L + 1$ and $M + 1$.

A $x^2 - 15x + 9 = 0$

B $x^2 - 11x + 9 = 0$

C $x^2 + 15x + 9 = 0$

D $x^2 - 15x + 8 = 0$

E $x^2 + 11x + 8 = 0$

Q5: If L and M are the roots of the equation $x^2 + 20x + 15 = 0$, what is the value of $\frac{1}{M} + \frac{1}{L}$?

A $-\frac{3}{4}$

B -35

C $-\frac{4}{3}$

D $\frac{4}{3}$

E 35

Q6: Given that L and M are the roots of the equation $3x^2 - 6x + 7 = 0$, find, in its simplest form, the quadratic equation whose roots are $L + M$ and LM.

A $3x^2 - 13x + 14 = 0$

B $3x^2 + 6x + 7 = 0$

C $3x^2 + 7x - 6 = 0$

D $3x^2 + 13x + 14 = 0$

E $3x^2 - 6x + 7 = 0$

Q7: If L and M are the roots of the equation $x^2 - 19x + 9 = 0$, find, in its simplest form, the quadratic equation whose roots are $L - 2$ and $M - 2$.

A $x^2 - 23x - 25 = 0$

B $x^2 - 23x + 32 = 0$

C $x^2 - 15x - 25 = 0$

D $x^2 - 15x + 32 = 0$

E $x^2 + 15x - 25 = 0$

Q8: Given that L and M are the roots of the equation $x^2 + x - 2 = 0$, find, in its simplest form, the quadratic equation whose roots are $L^2 + M$ and $M^2 + L$.

A $x^2 - x - 5 = 0$

B $x^2 + 4x - 5 = 0$

C $x^2 - 4x - 5 = 0$

D $x^2 - 4x + 9 = 0$

E $x^2 + x - 5 = 0$

Q9: Given that L and M are the roots of the equation $3x^2 + 16x - 1 = 0$, find, in its simplest form, the quadratic equation whose roots are $\frac{L}{2}$ and $\frac{M}{2}$.

A $12x^2 - 32x - 1 = 0$

B $x^2 + 32x - 1 = 0$

C $12x^2 + 32x + 1 = 0$

D $x^2 - 32x - 1 = 0$

E $12x^2 + 32x - 1 = 0$

1 - 4 The relation between two roots of the second degree equation and the cofficients of its terms

First: Complete each of the folowing:

1. if $x = 3$ is one of the roots of the equation $x^2 + M x - 27 = 0$, then M = ... and the other root is ....

2. If the product of the two roots of the equation : $2x^2 + 7x + 3K = 0$ equals the sum of the two roots of the equation: $x^2 - (K + 4)x = 0$, then K = ...

3. The quadratic equation which each of its two roots increases 1 than each of the two roots of the quadratic equation $x^2 - 3x + 2 = 0$ is ....

4. The quadratic equation which each of its two roots decreases 1 than each of the two roots of the quadratic equation $x^2 - 5x + 6 = 0$ is ....

Second: multiple choice

5. If one of the two roots of the equation $x^2 - 3x + c = 0$ is twice the other, then c= ....

   A) -4 B) -2 C) 2 D) 4

6. If one of the two roots of the equation $ax^2 - 3x + 2 = 0$ is the multiplicative inverse of the other , then a = ...

   A) $\frac{1}{3}$ B) $\frac{1}{2}$ C) 2 D) 3

7. If one of the two roots of the equation $x^2 - (b - 3)x + 5 = 0$ is the additive inverse of the other, then b = ...

   A) -5 B) -3 C) 3 D) 5

Third : Answer the following questions

8. Find the sum and the product of the two roots in each of the following equations:

   A) $3x^2 + 19x - 14 = 0$ B) $4x^2 + 4x - 35 = 0$

   ....

9. Find the value of a , then find the other root in each of the following equations:

   A) If: $x = -1$ is one of the two roots of the equation $x^2 - 2x + a = 0$ ....

   B) If: $x = 2$ is one of the two roots of the equation $ax^2 - 5x + a = 0$ ....

(10) Find the values of a and b if:

(A) 2 and 5 are the two roots of the equation $x^2 + a\,x + b = 0$ ....

(B) -3 and 7 are the two roots of the equation $ax^2 - b\,x - 21 = 0$ ....

(C) $-1$and $\frac{3}{2}$ are the two roots of the equation $a\,x^2 - x + b = 0$ ....

(D) $\sqrt{3}i$ and $-\sqrt{3}i$ are the two roots of the equation $x^2 + a\,x + b = 0$..............................

(11) Investigate the type of the two roots in each of the following equations, then find the solution set of each equation:

(A) $x^2 + 2x - 35 = 0$ ....

(B) $2x^2 + 3x + 7 = 0$ ....

(C) $x(x - 4) + 5 = 0$ ....

(D) $3x(3x - 8) + 16 = 0$ ....

(12) Find the value of c, if the two roots of the equation $c\,x^2 - 12x + 9 = 0$ are equal.

....

(13) Find the value of a, if the two roots of the equation $x^2 - 3x + 2 + \frac{1}{a} = 0$ are equal.

....

(14) Find the value of c, if the two roots of the equation $3x^2 - 5\,x + c = 0$ are equal, then find the two roots.

....

(15) Find the value of K, if one a root of the equation $x^2 + (K - 1)\,x - 3 = 0$ is the additive inverse to the other root.

....

(16) Find the value of K, if one root of the equation $4\,K\,x^2 + 7\,x + K^2 + 4 = 0$ is the multiplicative inverse to the other root.

....

(17) Form the quadratic equation whose two roots are :

(A) $-2, 4$ ....

(B) $-5\,i, 5\,i$ ....

(C) $\frac{2}{3}, \frac{3}{2}$ ....

(D) $1 - 3i, 1 + 3i$ ....

(E) $3 - 2\sqrt{2}i$, $3 + 2\sqrt{2}i$ ....

(18) Find the quadratic equation in which each of the two roots is twice one of the roots of the equation $2x^2 - 8x + 5 = 0$

....

(19) Find the quadratic equation in which each of the two roots exceeds 1 than one of the two roots of the equation : $x^2 - 7x - 9 = 0$

(20) Find the quadratic equation in which each of its two roots equals the square of the corresponding root of the equation : $x^2 + 3x - 5 = 0$

(21) If L and M are the two roots of the equation $x^2 - 7x + 3 = 0$, then find the quadratic equation whose roots are:

(A) $2L, 2M$ (B) $L + 2, M + 2$ (C) $\frac{2}{L}, \frac{2}{M}$ (D) $L + M, LM$

Lesson (5) — The sign of the function

Lesson objectives

Related Links

- Investigate the sign of constant function.
- Investigate the sign of linear function.
- Investigate the sign of quadratic function.

First: The constant function $F(x) = c$ where $c \neq 0$

So, the sign $f(x)$ is like the sign of c. for all $x \in \mathbb{R}$

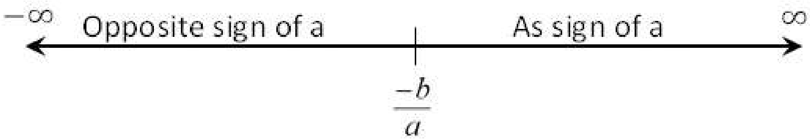
For example:

(1) $F(x) = 7$ $\therefore f(x)$ is +ve for all $x \in \mathbb{R}$

(2) $F(x) = -3$ $\therefore f(x)$ is –ve for all $x \in \mathbb{R}$

Second: The linear function $F(x) = a\,x + b$ where $a \neq 0$

Let $a\,x + b = 0$, then $x = \frac{-b}{a}$



Example ❶

Determine the sign the function:

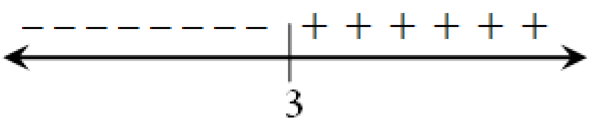
① $F(x) = 2\,x - 6$ **② $F(x) = 1 - 2x$**

Solution

① $F(x) = 2\,x - 6$

Put: $2\,x - 6 = 0$

$\therefore 2x = 6$ $\therefore x = 3$

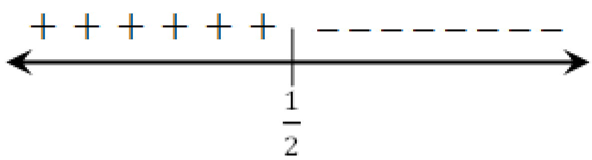


$$f(x)\begin{cases} +\text{ve} & \text{at } x \in\,]3\,,\infty[\\ \text{zero} & \text{at } x = 3 \\ -\text{ve} & \text{at } x \in\,]-\infty\,, 3[\end{cases}$$

② $F(x) = 1 - 2x$

Put: $1 - 2x = 0$

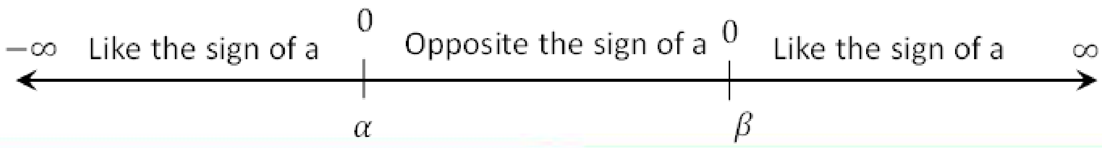
$\therefore 2x = 1$ $\therefore x = \frac{1}{2}$

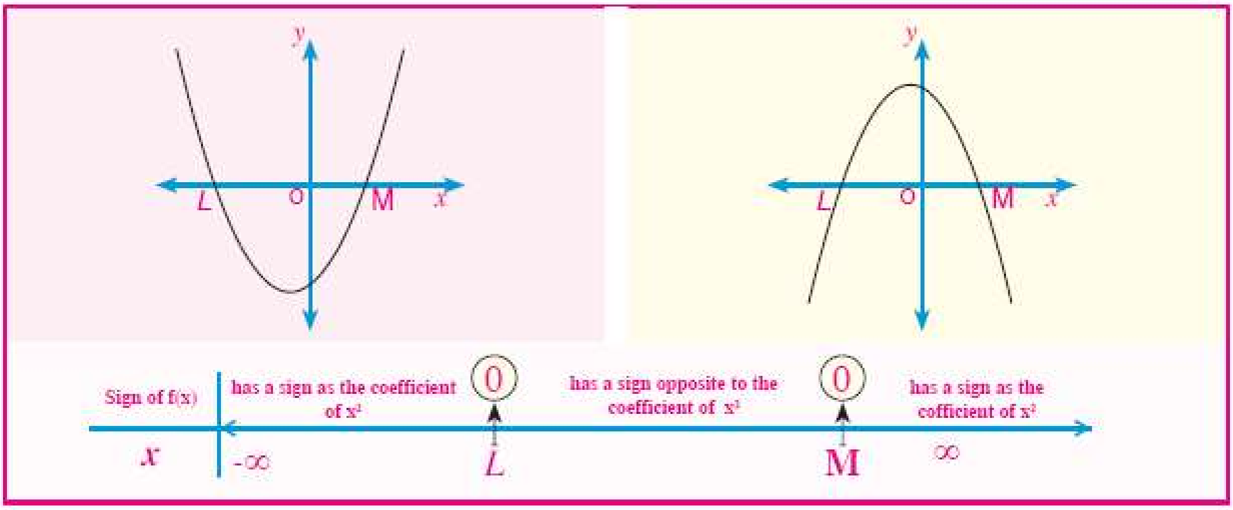


$$f(x)\begin{cases} +\text{ve} & \text{at } x \in\,]-\infty\,, \frac{1}{2}[\\ \text{zero} & \text{at } x = \frac{1}{2} \\ -\text{ve} & \text{at } x \in\,]\frac{1}{2}\,, \infty[\end{cases}$$

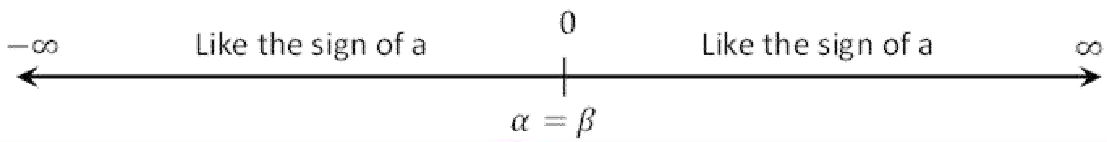
Third: The quadratic function $F(x) = ax^2 + b\,x + c$

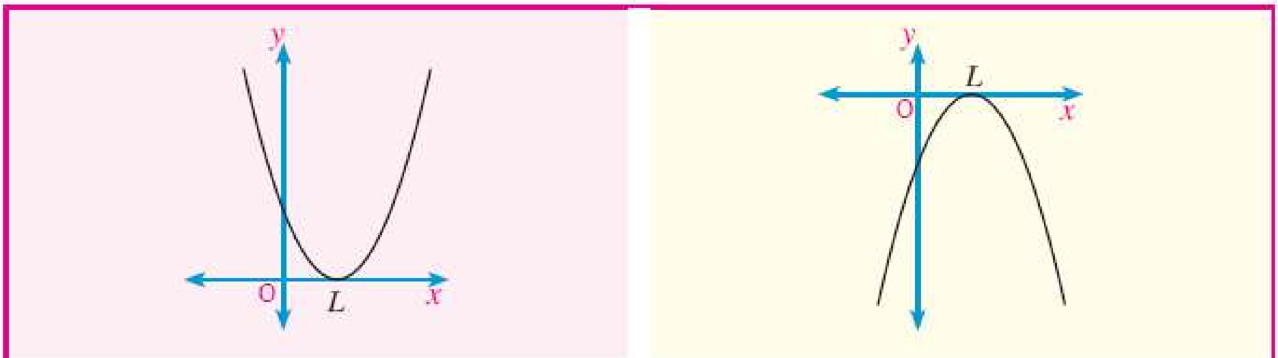
Case (1): **If:** $b^2 - 4\,a\,c > 0$ (two different real roots) $\{\alpha , \beta\}$





Case (2): **If:** $b^2 - 4\,ac = 0$ (equal roots) $\{\alpha\}$



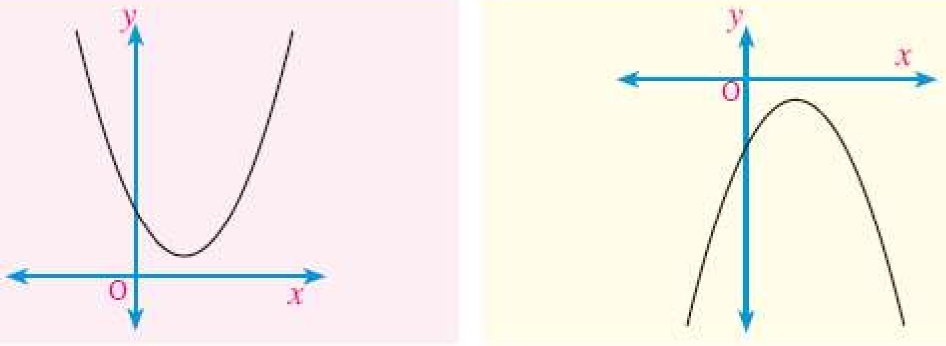


If: $a > 0$
$f(x) > 0 \ \ \forall \ x \neq L,$
$f(x) = 0$ when $x = L$

If: $a < 0$
$f(x) < 0 \ \ \forall \ x \neq L,$
$f(x) = 0$ when $x = L$

Case (3): **If:** $b^2 - 4ac < 0$ (no real roots)

$-\infty$ Like the sign of a ∞



If: $a > 0$ | If: $a < 0$
$f(x) > 0 \ \forall x \in \mathbb{R}$ | $f(x) < 0 \ \forall x \in \mathbb{R}$

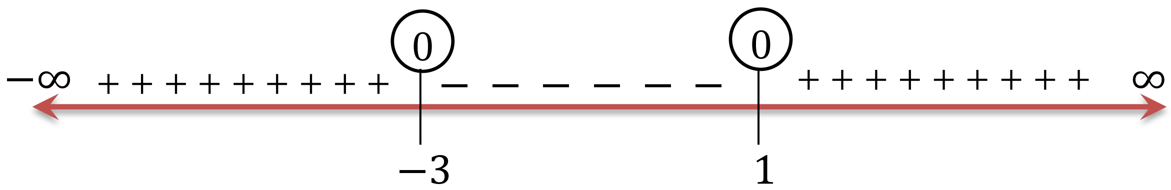
Example ❷

Determine the sign the function:

① $F(x) = x^2 + 2x - 3$

Solution

Put: $x^2 + 2x - 3 = 0$ $x = 1$ or $x = -3$



$$f(x) \begin{cases} +\text{ ve} & \text{at } x \in \mathbb{R} - [-3, 1] \\ \text{zero} & \text{at } x \in \{-3, 1\} \\ -\text{ ve} & \text{at } x \in]-3, 1[\end{cases}$$

② $F(x) = x^2 - 3x + 5$

Solution

Put: $x^2 - 3x + 5 = 0$ (***two comples numbers***)

$-\infty$ + ∞

$f(x)$ is **+ ve** for every $x \in \mathbb{R}$

---

③ $F(x) = 4x^2 - 12x + 9$

Solution

Put: $4x^2 - 12x + 9 = 0$ $\therefore x = \frac{3}{2}$

$-\infty$ + + + + + + + + + + + + + + 0 + + + + + + + + + + + + + + ∞

$\frac{3}{2}$

$$f(x) \begin{cases} +\text{ve} & \text{at } x \in \mathbb{R} - \left\{\frac{3}{2}\right\} \\ \text{zero} & \text{at } x = \frac{3}{2} \end{cases}$$

PRACTICE

Q1: In which of the following intervals is $f(x) = -8$ negative?

A $]-8, 8[$

B $]-\infty, \infty[$

C $]-\infty, 8[$

D $]-8, \infty[$

E $]8, \infty[$

Q2: For which values of x is the function $f(x) = 8x - 13$ positive?

A $x \leq \frac{13}{8}$

B $x > -\frac{13}{8}$

C $x > \frac{13}{8}$

D $x \geq \frac{13}{8}$

E $x < \frac{13}{8}$

Q3: Determine the sign of the function $f(x) = (x-8)(x-7)$.

A The function is positive when $x \in \mathbb{R} - [7,8]$, the function is negative when $x \in]7,8[$, and the function equals zero when $x \in \{7,8\}$.

B The function is positive when $x \in \mathbb{R} - \{7,8\}$, the function is negative when $x \in [7,8]$, and the function equals zero when $x \in \{7,8\}$.

C The function is positive when $x \in [7,8]$, the function is negative when $x \in \mathbb{R} - \{7,8\}$, and the function equals zero when $x \in \{7,8\}$.

D The function is positive when $x \in]7,8[$, the function is negative when $x \in \mathbb{R} - [7,8]$, and the function equals zero when $x \in \{7,8\}$.

Q4: Determine the sign of the function $f(x) = x^2 - 16x + 64$.

A The function is positive when $x \in \mathbb{R} - \{8\}$, and the function equals zero when $x \in \{-8, 8\}$.

B The function is positive for all $x \in \mathbb{R}$.

C The function is positive when $x \in \mathbb{R} - \{-8\}$, and the function equals zero when $x = -8$.

D The function is positive when $x \in \mathbb{R} - \{8\}$, and the function equals zero when $x = 8$.

E The function is positive when $x \in \mathbb{R} - \{-8, 8\}$, and the function equals zero when $x \in \{-8, 8\}$.

Q5: Determine the sign of the function $f(x) = -x^2 - 2x - 7$.

A. The function is positive for all $x \in \mathbb{R} - \{0\}$, and the function equals zero when $x = 0$.

B. The function is negative for all $x \in \mathbb{R} - \{0\}$, and the function equals zero when $x = 0$.

C. The function is positive for all $x \in \mathbb{R}$.

D. The function is negative for all $x \in \mathbb{R}$.

Q6: Determine the interval in which the function $f(x) = -15 - 8x - x^2$ is NOT negative.

A. $\mathbb{R} - [-5, -3]$

B. $[-5, -3]$

C. $[3, 5]$

D. $\mathbb{R} - [3, 5]$

E. $\mathbb{R}$

Q7: What are the values of x for which the functions $f(x) = x - 5$ and $g(x) = x^2 + 2x - 48$ are both positive?

A. $x > -8$

B. $x > 6$

C. $x < -8$

D. $x < 6$

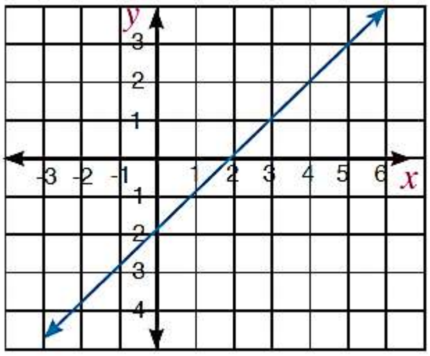
E. $x > 5$

Sign of a Function

1 - 5

First : complete each of the following:

1. The sign of the function f, where $f(x) = -5$ is .... in the interval ....
2. The sign of the function f, where $f(x) = x^2 + 1$ is .... in the interval ....
3. The sign of the function f, where $f(x) = x^2 - 6x + 9$ is positive in the interval ....
4. The sign of the function f, where $f(x) = x - 2$ is positive in the interval ....
5. The sign of the function f, where $f(x) = 3 - x$ is negative in the interval ....
6. The sign of the function f, where $f(x) = -(x-1)(x+2)$ is positive in the interval ....
7. The sign of the function f, where $f(x) = x^2 + 4x - 5$ is negative in the interval ....
8. The figure opposite represents a first degree function in x:
   - **A** The function is positive in the interval ....
   - **B** the function is negative in the interval ....

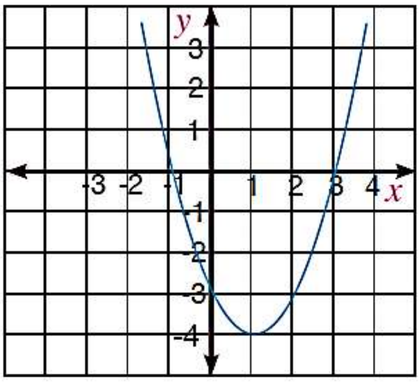


9) The figure opposite represents a second degree function in x:

A) $f(x) = 0$ when $x \in$ ....

B) $f(x)\ 0 >$ when $x \in$ ....

C) $f(x)\ 0 <$ when $x \in$ ....



Second: answer the following questions:

10) In exercises from (A) to (N), determine the sign of each of the following functions:

A) $f(x) = 2$ ....

B) $f(x) = 2x$ ....

C) $f(x) = -3x$ ....

D) $f(x) = 2x+4$ ....

E) $f(x) = 3 - 2x$ ....

F) $f(x) = x^2$ ....

G) $f(x) = 2x^2$ ....

H) $f(x) = x^2 - 4$ ....

I) $f(x) = 1 - x^2$ ....

J) $f(x) = (x - 2)(x + 3)$ ....

K) $f(x) = (2x - 3)^2$ ....

L) $f(x) = x^2 - x - 2$ ....

M) $f(x) = x^2 - 8x + 16$ ....

N) $f(x) = -4x^2 + 10x - 25$ ....

11) Graph the curve of the function $f(x) = x^2 - 9$ in the interval $[-3, 4]$, hence determine the sign of $f(x)$.

12) Graph the curve of the function $f(x) = -x^2 + 2x + 4$ in the interval $[-3, 5]$, hence determine the sign of $f(x)$.

Lesson (6) | Quadratic Inequalities in one variable

Lesson objectives

Related Links

- Solve the quadratic inequality in one variable.

Solving the quadratic inequality in one variable

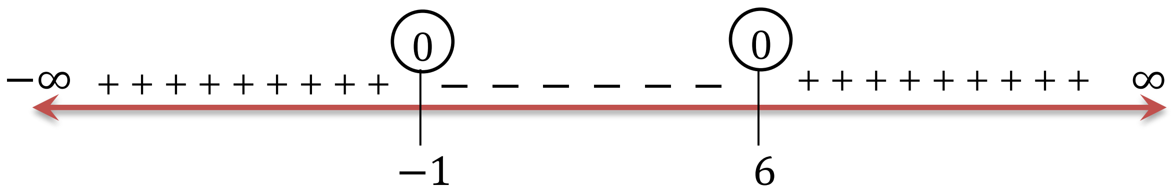
Example ❶

Solve the inequality: $x^2 - 5x - 6 > 0$

Solution

$\because x^2 - 5x - 6 > 0$

Put: $x^2 - 5x - 6 = 0$ $\qquad x = 6 \qquad$ or $\qquad x = -1$



$\therefore$ The S.S. = $\mathbb{R} - [-1, 6]$ or The S.S. = $]-\infty, -1[\cup]6, \infty[$

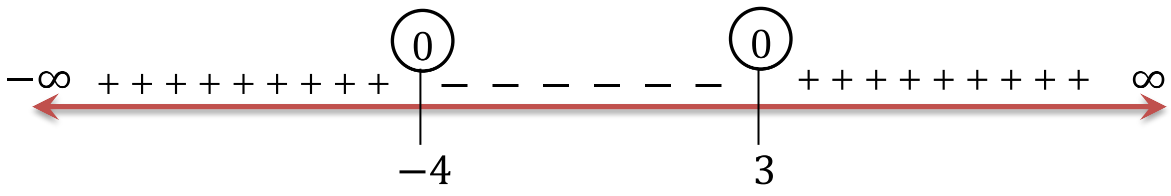
Example ❷

Solve the inequality: $x^2 + 2x - 8 \geq 0$

Solution

$\because x^2 + 2x - 8 \geq 0$

Put: $x^2 + 2x - 8 = 0$ $\qquad x = 2 \qquad$ or $\qquad x = -4$



$\therefore$ The S.S. = $\mathbb{R} -]-4, 3[$ or The S.S. = $]-\infty, -4] \cup [3, \infty[$

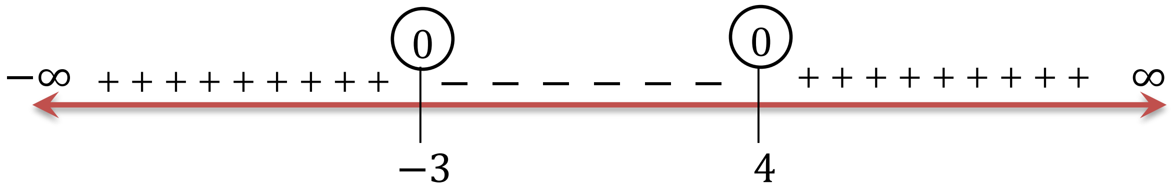
Example ❸

Solve the inequality: $x^2 - x - 12 < 0$

Solution

$\because x^2 - x - 12 < 0$

Put: $x^2 - x - 12 = 0$ $\qquad x = 4$ $\qquad$ or $\qquad x = -3$



$\therefore$ The S.S. $=]-3, 4[$

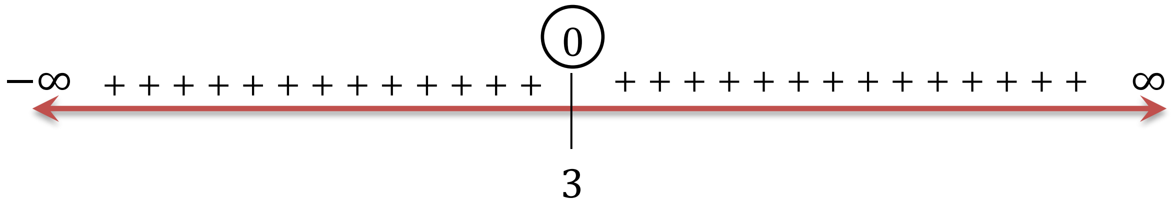
Example ❹

Solve the inequality: $x^2 - 6x < -9$

Solution

$\because x^2 - 6x < -9$ $\qquad \therefore x^2 - 6x + 9 < 0$

Put: $x^2 - 6x + 9 = 0$ $\qquad x = 3$



$\therefore$ The S.S. $= \emptyset$

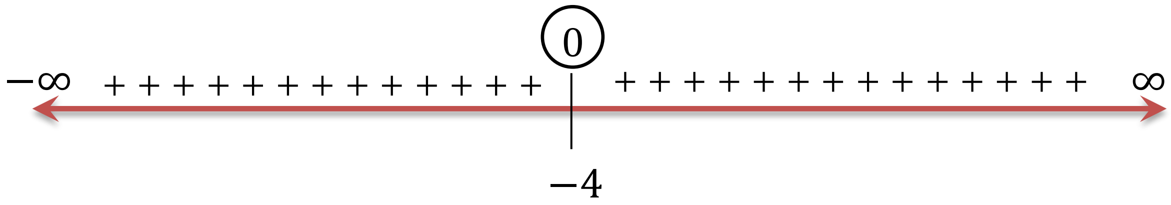
Example ❺

Solve the inequality: $x^2 + 8x > -16$

Solution

$\because x^2 + 8x > -16$ $\qquad \therefore x^2 + 8x + 16 > 0$

Put: $x^2 + 8x + 16 = 0$ $\qquad x = -4$



$\therefore$ The S.S. $= \mathbb{R} - \{-4\}$

PRACTICE

Q1: Solve $x^2 - 2x < 0$ graphically.

A $R -]0,2[$

B $\{0,2\}$

C $[0,2]$

D $]0,2[$

E $R - [0,2]$

Q2: Solve $x^2 - x - 6 < 0$.

A $[-2,3]$

B $R -]-2,3[$

C $R - [-2,3]$

D $\{-2,3\}$

E $]-2,3[$

Q3: Solve $5(x-1) - x(7-x) \leq x^2$ graphically.

A $]-2.5,\infty[$

B $]0,2.5[$

C $[-2.5,\infty[$

D $]-\infty,2.5]$

E $[-2.5,0[$

Q4: Solve $x^2 - x - 6 > 0$ graphically.

A $[-2, 3]$

B $\{-2, 3\}$

C $R -]-2, 3[$

D $]-2, 3[$

E $R - [-2, 3]$

Q5: Solve $-x^2 + 9 > 0$ graphically.

A $[-3, 3]$

B $R - [-3, 3]$

C $]-3, 3[$

D $\{-3, 3\}$

E $R -]-3, 3[$

Q6: Solve $(2 - 3x) - (x - 1) \geq -4 - (x - 2)^2$ graphically.

A $]-\infty, 4 - \sqrt{5}] \cup]4 + \sqrt{5}, \infty[$

B $]-\infty, 4 - \sqrt{5}] \cup [4 + \sqrt{5}, \infty[$

C $]-\infty, 4 - \sqrt{5}[\cup]4 + \sqrt{5}, \infty[$

D $]-\infty, 4 - \sqrt{5}[$

E $]4 + \sqrt{5}, \infty[$

Q7: Solve $2x^2 < 3x + 5$ graphically.

A $\left]-1, \frac{5}{2}\right[$

B $R - \left[-1, \frac{5}{2}\right]$

C $\left[-1, \frac{5}{2}\right]$

D $R - \left]-1, \frac{5}{2}\right[$

E $\left\{-1, \frac{5}{2}\right\}$

Q8: Solve $(2 - x)(x - 1) \geq 2 - (x - 1)^2$ graphically.

A $]3, \infty[$

B $[1, \infty[$

C $]-\infty, 3[$

D $[3, \infty[$

E $]-\infty, 3]$

Q9: Find all solutions to the inequality $x^2 + 121 \leq 0$. Write your answer as an interval.

A $[-11, 11]$

B $]-11, 11[$

C $\mathbb{R} -]-11, 11[$

D $\varnothing$

E $\mathbb{R} - [-11, 11]$

Quadratic inequalities

1 - 6

Find the solution set of each of the following quadratic inequalities:

1. $x^2 \leqslant 9$

2. $x^2 - 1 \leqslant 0$

3. $2x - x^2 < 0$

4. $x^2 + 5 \leqslant 1$

5. $(x - 2)(x - 5) < 0$

6. $x(x + 2) - 3 \leqslant 0$

7. $(x - 2)^2 \leqslant - 5$

8. $5 - 2x \leqslant x^2$

9. $x^2 \geqslant 6x - 9$

10. $3x^2 \leqslant 11x + 4$

11. $x^2 - 4x + 4 \geqslant 0$

UNIT TEST

First : choose the correct answer from the given answers:

1. The solution set of the equation $x^2 - 6x + 9 = 0$ in $\mathbb{R}$ is: ....
   (A) $\{-3\}$ (B) $\{3\}$ (C) $\{-3, 3\}$ (D) ϕ

2. The solution set of the equation $x^2 + 4 = 0$ is : ....
   (A) $\{-2\}$ (B) $\{2\}$ (C) $\{-2, 2\}$ (D) $\{-2i, 2i\}$

3. The simplest form of the expression $(1 - i)^4$ is : ....
   (A) -4 (B) 4 (C) $-4i$ (D) $4i$

4. If the two roots of the equation $x^2 - 4x + K = 0$ are real and different , then: ....
   (A) $K > 4$ (B) $K < 4$ (C) $K = 4$ (D) $K \geqslant 4$

5. If the two roots of the equation $x^2 - 12x + M = 0$ are equal, then M equals: ....
   (A) -36 (B) -6 (C) 6 (D) 36

6. The quadratic equation whose roots are $2 - 3i$ and $2 + 3i$ is : ....
   (A) $x^2 + 4x + 13 = 0$ (B) $x^2 - 4x + 13 = 0$ (C) $x^2 + 4x - 13 = 0$ (D) $x^2 - 4x - 13 = 0$

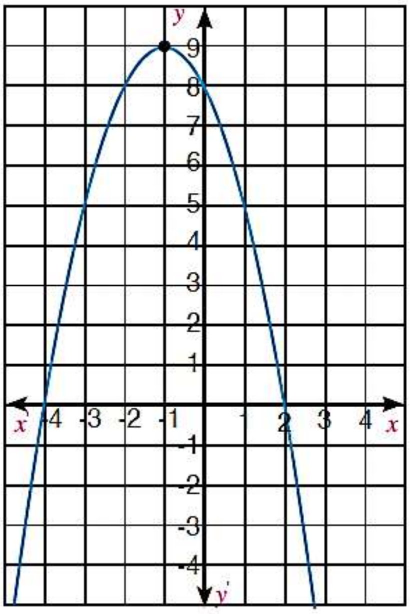
7. If $f : [-2, 4] \longrightarrow \mathbb{R}$ where $f(x) = 2 - x$, then the sign of the function f is negative in the interval:
   (A) $[-2, 2[$ (B) $[-2, 2]$ (C) $[2, 4]$ (D) $]2, 4]$

8. If one of the two roots of the equation $x^2 - (M + 2)x + 3 = 0$ is the additive inverse of the other, then M equals:
   (A) -3 (B) -2 (C) 2 (D) 3

9. If one of the two roots of the equation $2x^2 + 7x + K = 0$ is the multiplicative inverse of the other, then K equals:
   (A) -7 (B) -2 (C) 2 (D) 7

10. The solution set of the inequality $x^2 + x - 2 < 0$ is :
   (A) $]-2, 1[$ (B) $[-2, 1]$ (C) $\mathbb{R} - [-2, 1]$ (D) $\mathbb{R} -]-2, 1[$

Second : the figure opposite represents the graph of a quadratic function



(11) Complete each of the following :

A) The range of the function f is ....

B) The maximum value of the function f equals ....

C) Type of the two roots is ....

D) The solution set of the equation $f(x) = 0$ is ....

E) $f(x) > 0$ when $x \in$ ....

F) $f(x) < 0$ when $x \in$ ....

G) $f(x) = 0$ when $x =$ ....

(12) Write the rule of the quadratic function which passes through $(-4, 0)$, $(2, 0)$ and $(-1, 9)$.

....

Third: Answer the following questions

(13) Determine the type of roots of each of the following equations, then find the solution set of each equation .

A) $x^2 - 2x = 0$ ....

B) $(x - 1)^2 = 4$ ....

C) $x^2 - 6x + 9 = 0$ ....

D) $x^2 + 3x - 28 = 0$ ....

E) $6x(x - 1) = 6 - x$ ....

(14) Use the general formula to solve the following equations approximating the result to the nearest hundredth.

A) $x^2 + 4x + 2 = 0$ ....

B) $x^2 - 3(x - 2) = 5$ ....

(15) Find the solution set of the following equations in the set of complex numbers .

A) $x^2 + 9 = 0$

B) $x^2 + 2x + 2 = 0$

C) $x^2 + 4x + 5 = 0$

....

16 Find the values of a and b in each of the following :

A $(7 - 3i) - (2 + i) = a + b\,i$

B $(2 - 5i)(3 + i) = a + b\,i$

C $\frac{10}{2 + i} = a + b\,i$

D $\frac{6 - 4i}{1 - i} = a + b\,i$

17 Find the value of M in each of the following :

A If the two roots of the equation $2x^2 + M\,x + 18 = 0$ are equal ....

B If one of the two roots of the equation $x^2 + 3\,x + M = 0$ is twice the other root ....

18 Investigate the sign of the function f in each of the following :

A $f(x) = x^2 - 2\,x - 8$

B $f(x) = 4 - 3x - x^2$

....

....

19 Find the solution set of each of the following inequalities :

A $x^2 - x - 12 > 0$

B $x^2 - 7x + 10 \leqslant 0$

....

....

Geometry

Futures Language Schools
Mathematical Department

First Secondary

First Term

Student Name:

Unit 2

Lesson (1) Similarity of Polygons

Lesson objectives

Related Links

- Identify the concept of similarity.
- Classify Similarity of polygons.
- Identify drawing Scale.
- Identify Golden rectangle and golden ratio.

Definition

Two polygons P1& P2 (same number of sides) are said to be similar if:

1) The corresponding angles are equal in measure.

2) The corresponding sides are proportional.

Notes:

1) Any two congruent polygons are similar.

2) Any two regular polygons (*same number of sides*) are similar.

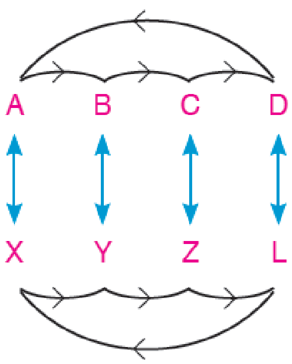
[Any two squares are similar – any two equilateral triangles are similar]

Remark

If polygon ABCD ~ polygon XYZL then:

① $\angle A \equiv \angle X$, $\angle B \equiv \angle Y$, $\angle C \equiv \angle Z$, $\angle D \equiv \angle L$

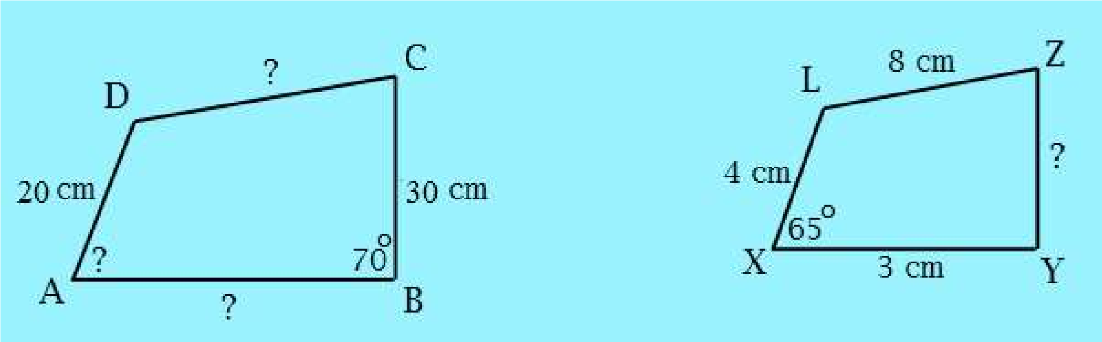
② $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZL} = \frac{DA}{LX} = k$ (similarity ratio), $k \neq 0$



The scale factor of similarity of polygon ABCD to polygon XYZL equals k, and scale factor of similarity of polygon XYZL to polygon ABCD equals $\frac{1}{k}$

Example ❶

The two polygons ABCD & XYZL are similar find the missing elements?



Solution

$\because$ polygon ABCD ~ polygon XYZL

$\therefore$ m (∠ A) = m (∠ X) = 65°

$\because \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZL} = \frac{DA}{LX}$ $\therefore \frac{AB}{3} = \frac{30}{YZ} = \frac{CD}{8} = \frac{20}{4}$

$\therefore$ AB = 15 cm , YZ = 6 cm & CD = 40 cm

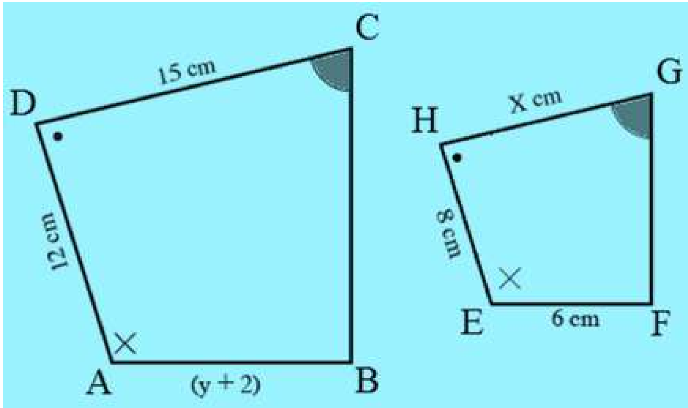
Example ❷

In the figure opposite:

polygon ABCD ~ polygon EFGH.

① **Find** the scale factor of similarity of polygon ABCD to polygon EFGH.

② **Find** the values of x and y.



Solution

$\because$ polygon ABCD ~ polygon EFGH

$\therefore$ The scale factor o similarity = $\frac{DA}{HE} = \frac{12}{8} = \frac{3}{2}$

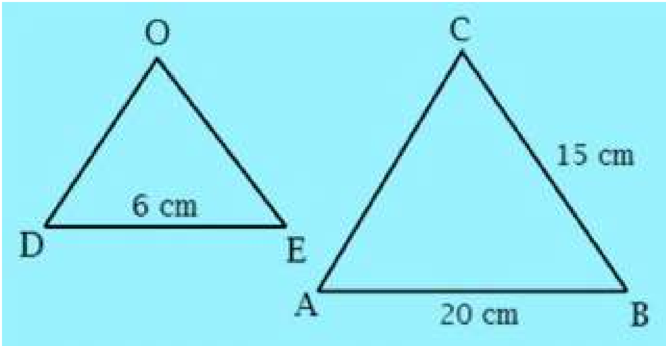
$\because \frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH} = \frac{DA}{HE}$ $\therefore \frac{(y+2)}{6} = \frac{15}{x} = \frac{12}{8}$

$\therefore x = 10$ cm

$\therefore y + 2 = 9$ $\therefore y = 7$ cm

Example ❸

In the opposite figure:



m (∠ A) = m (∠ D), m (∠ B) = m (∠ E)

AB = 20 cm, BC = 15 cm and DE = 6 cm

① **Prove that** Δ ABC ~ Δ DEO

② **Find the length of** $\overline{EO}$

Solution

In Δ ABC & Δ DEO:

∵ m (∠ A) = m (∠ D) & m (∠ B) = m (∠ E) ∴ m (∠ C) = m (∠ O)

∴ Δ ABC ~ Δ DEO ∴ $\frac{AB}{DE} = \frac{BC}{EO} = \frac{CA}{OD}$

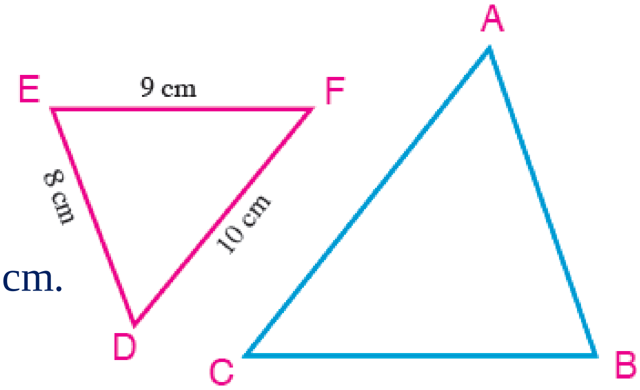
∴ $\frac{20}{6} = \frac{15}{EO}$ ∴ EO = 4.5 cm

Notice that:

If polygon M1 ~ polygon M2, then $\frac{\text{perimeter of M1}}{\text{perimeter of M2}}$ = Similarity ratio (*scale factor*)

Example ❹

In the figure opposite:



Δ ABC ~ Δ DEF, DE = 8 cm, EF = 9 cm,

FD = 10 cm. If the perimeter of Δ ABC = 81cm.

Find the side lengths of Δ ABC

Solution

∵ Δ ABC ~ Δ DEF ∴ $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{\text{Perimeter of Δ ABC}}{\text{Perimeter of Δ DEF}}$

∴ $\frac{AB}{8} = \frac{BC}{9} = \frac{CA}{10} = \frac{81}{27}$

∴ AB = 24 cm , BC = 27 cm & CA = 30 cm

Similarity ratio of two polygons

Let K be the similarity ratio of polygon M_1 to polygon M_2

If: $K > 1$ **then** polygon M_1 is an **enlargement** of polygon M_2

$0 < K < 1$ **then** polygon M_1 is a **shrinking** of polygon M_2

$K = 1$ **then** polygon M_1 is **congruent** to polygon M_2

In general: *you can use the similarity ratio in calculation of the dimensions of similar figures.*

Example ❺

ABCD is a rectangle in which AB = 5 cm and BC = 8 cm. Find the dimensions of another rectangle similar to it, if:

① Scale factor = 1.4 **② Scale factor = 0.6**

Solution

Let the rectangle XYZL ~ the rectangle ABCD

$\therefore \frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZL}{CD} = \frac{XL}{AD} = \text{scale factor}$

① If the scale factor = 1.4 $\therefore \frac{XY}{AB} = \frac{YZ}{BC} = 1.4$ $\therefore \frac{XY}{5} = \frac{YZ}{8} = 1.4$

$\therefore$ XY = 7 cm & YZ = 8.4 cm **[*Enlargement*]**

② If the scale factor = 0.6 $\therefore \frac{XY}{AB} = \frac{YZ}{BC} = 0.6$ $\therefore \frac{XY}{5} = \frac{YZ}{8} = 0.6$

$\therefore$ XY = 3 cm & YZ = 4.8 cm **[*Shrinking*]**

PRACTICE

Q1: A polygon has sides 2, 4, 3, 8, and 4. A second similar polygon has perimeter 31.5. What are its sides?

A 1.3 cm, 2.7 cm, 2 cm, 5.3 cm, 2.7 cm

B 3.5 cm, 5.5 cm, 4.5 cm, 9.5 cm, 5.5 cm

C 4 cm, 5 cm, 4.5 cm, 13 cm, 5 cm

D 3 cm, 6 cm, 4.5 cm, 12 cm, 6 cm

Q2: If $ABCDE \sim PQRST$, find the scale factor of $ABCDE$ to $PQRST$ and the perimeter of $PQRST$.

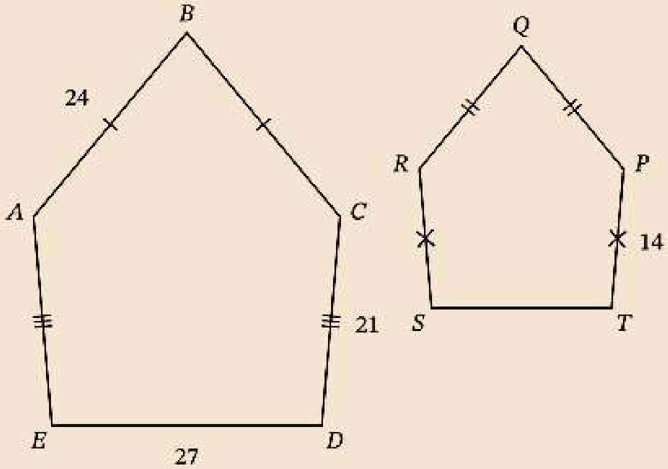
A The scale factor is $\frac{3}{2}$, and the perimeter is 78.

B The scale factor is $\frac{27}{14}$, and the perimeter is 117.

C The scale factor is $\frac{2}{3}$, and the perimeter is 64.

D The scale factor is $\frac{14}{27}$, and the perimeter is 60.7.

E The scale factor is $\frac{3}{2}$, and the perimeter is 117 .



Q3: If $BGCD \sim LYNZ$, then $\frac{BG}{CD} = \frac{\dots}{NZ}$.

A BG

B LY

C YN

D ZL

E GC

Q4: A rectangle that is 15 by 10 is similar to a second rectangle with perimeter 40. Find the length and the area of the second rectangle.

A length = 15, area = 150

B length = 12, area = 96

C length = 8 , area = 96

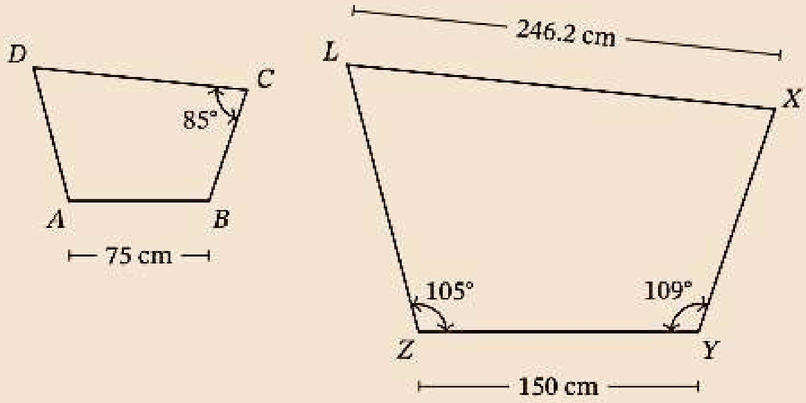
Q5: Given that $ABCD \sim ZYXL$, find $m\angle XLZ$ and the length of $\overline{CD}$.

A $m\angle XLZ = 105°$, $CD = 123.1$ cm

B $m\angle XLZ = 61°$, $CD = 123.1$ cm

C $m\angle XLZ = 61°$, $CD = 7.5$ cm

D $m\angle XLZ = 109°$, $CD = 7.5$ cm



Q6: Which of the following statements correctly defines similarity for polygons?

A Two polygons are said to be similar if their corresponding sides are equal.

B Two polygons are said to be similar if their corresponding angles are complementary and their corresponding sides are equal.

C Two polygons are said to be similar if their corresponding sides are congruent.

D Two polygons are said to be similar if their corresponding angles are equal.

E Two polygons are said to be similar if their corresponding angles are congruent and their corresponding sides are in proportion.

Q7: $ABCD \sim ZYXL$ and the perimeter of $ABCD = 177$ cm. Calculate the scale factor of similarity of $ZYXL$ to $ABCD$ and the perimeter of $ZYXL$.

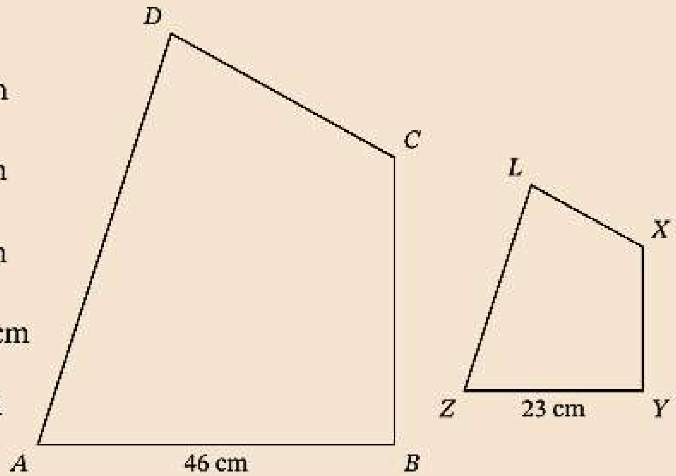
A scale factor $= \frac{1}{2}$, perimeter of $ZYXL = 88.5$ cm

B scale factor $= \frac{1}{4}$, perimeter of $ZYXL = 708$ cm

C scale factor $= \frac{1}{2}$, perimeter of $ZYXL = 354$ cm

D scale factor $= \frac{1}{4}$, perimeter of $ZYXL = 44.25$ cm

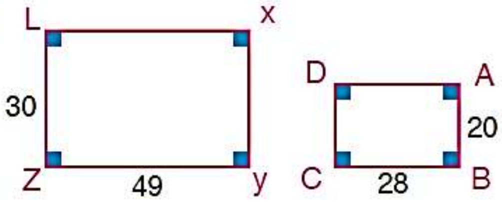
E scale factor $= 2$, perimeter of $ZYXL = 354$ cm



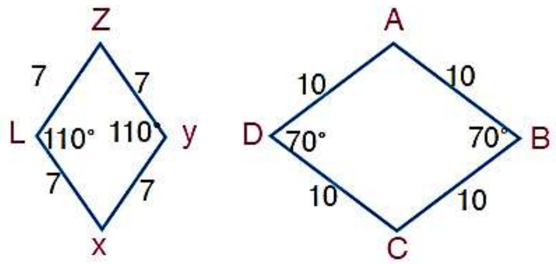
2 - 1 Similarity of Polygons

1 Show which of the following pairs of polygons are similar, write the similar polygons in order of their corresponding vertices and determine the scale factor of similarity (side lengths are estimated in centimetres).

A



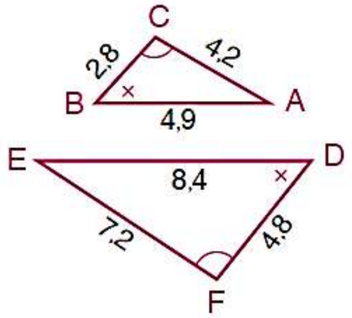
B



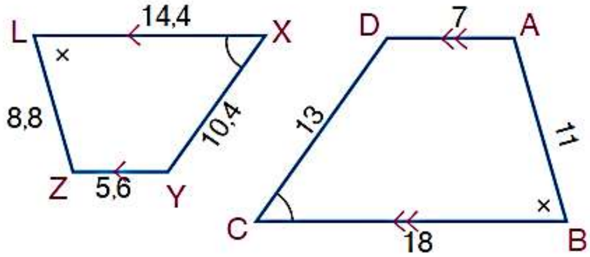
....

....

C



D



....

....

2 If polygon ABCD ~ polygon XYZL, complete:

A $\frac{AB}{BC} = \frac{\cdots\cdots}{YZ}$

B $AB \times ZL = XY \times$ ....

C $\frac{BC + YZ}{YZ} = \frac{\cdots\cdots + LX}{LX}$

D $\frac{\text{Perimeter of polygon } \cdots\cdots}{\text{Perimeter of polygon } \cdots\cdots} = \frac{XY}{AB}$

3 Polygon ABCD ~ polygon XYZL, If $AB = 32$cm, $BC = 40$ cm, $XY = 3m - 1$ and $YZ = 3m + 1$, Find the numerical value of m. ....

4 The dimensions of a rectangle are 10cm and 6cm. Find the perimeter and the area of another rectangle similar to it if:

A Scale factor equals 3.

B Scale factor equals 0,4

Lesson (2)
Similarity of Triangles

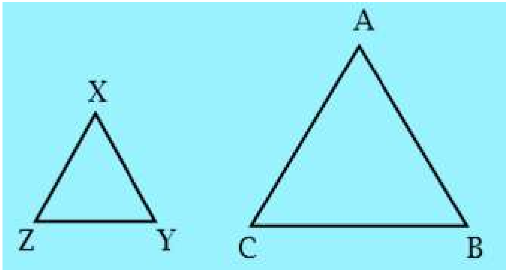
Lesson objectives

Related Links

- Classify Cases of similarity of triangles.
- Identify the properties of the perpendicular drawn from the vertex of the right angle to the hypotenuse of the right angled triangle.

First case:

Two triangles are similar if the measures of the corresponding angles of two triangles are equal.



If m (∠ A) = m (∠ X)

m (∠ B) = m (∠ Y)

m (∠ E) = m (∠ Z)

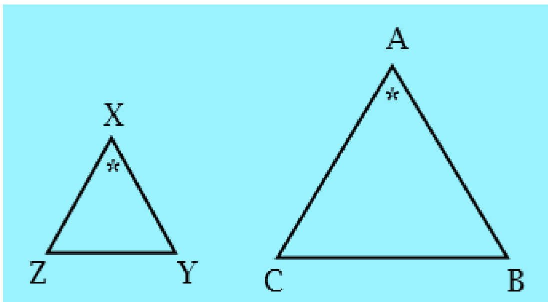
∴ Δ ABC ~ Δ XYZ

Special cases:

1) Any two congruent triangles are similar.
2) Any two isosceles triangles are similar it the measure of one the base angles in one of them is equal to the measure of the base angle of other.
3) Two right angle triangles are similar if one of the acute angles in one is equal to the measure of a cote angle of other.

The Second Case:

Two triangles are similar if the measure of an angle of a triangle equals the measure of an angle of another triangle, and the lengths of the sides enclosing by these two angles are proportional.



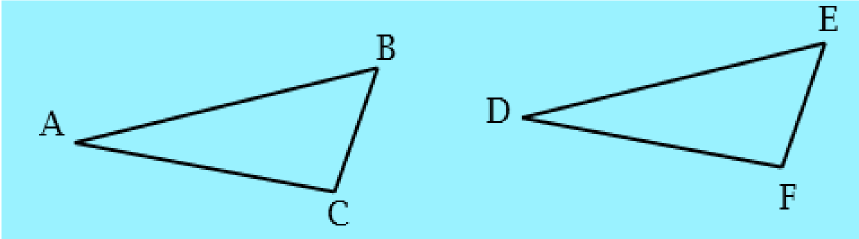
If m (∠ A) = m (∠ X)

And $\frac{AB}{XY} = \frac{AC}{XZ}$

∴ Δ ABC ~ Δ XYZ

The Third Case:

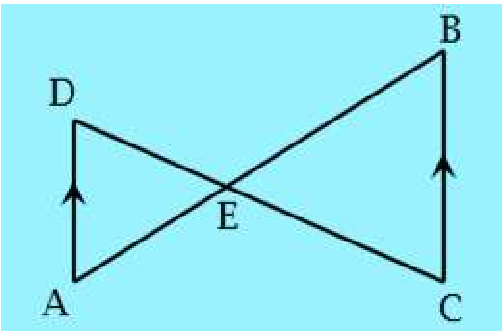
Two triangles are similar if the corresponding sides of two triangles are proportional.



If $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

$\therefore \Delta\ ABC \sim \Delta\ DEF$

Example ❶



In the opposite figure:

$\overline{AD} \parallel \overline{CB}$ **Prove that:**

(1) $\Delta\ AED \sim \Delta\ BEC$

(2) AE . EC = DE . EB

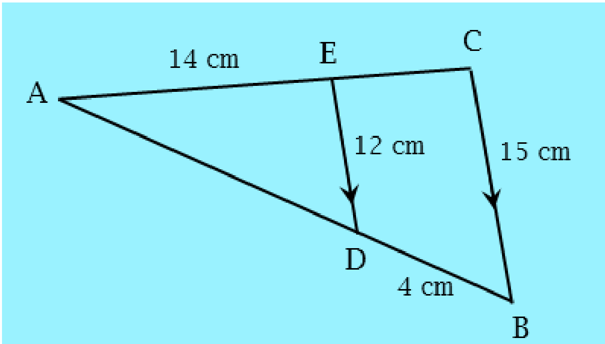
Solution

In Δ AED and Δ BEC:

$\because\ \overline{AD} \parallel \overline{CB}$ $\therefore\ \Delta\ AED \sim \Delta\ BEC$ **⟶ (1)**

$\therefore \frac{AE}{BE} = \frac{ED}{EC} = \frac{AD}{BC}$ $\therefore$ AE . EC = DE . EB **⟶ (2)**

Example ❷



In the opposite figure:

ABC is a triangle in which:

$\overline{ED} \parallel \overline{CB}$, AE = 14 cm, ED = 12 cm,

BD = 4 cm and BC = 15 cm

Find the length of each of $\overline{AC}$, $\overline{AD}$ and $\overline{AB}$

Solution

In Δ ABC and Δ ADE:

$\because\ \overline{BC} \parallel \overline{DE}$ $\therefore\ \Delta\ ABC \sim \Delta\ ADE$

$\therefore \frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE}$ $\therefore \frac{AD+4}{AD} = \frac{15}{12} = \frac{AC}{14}$

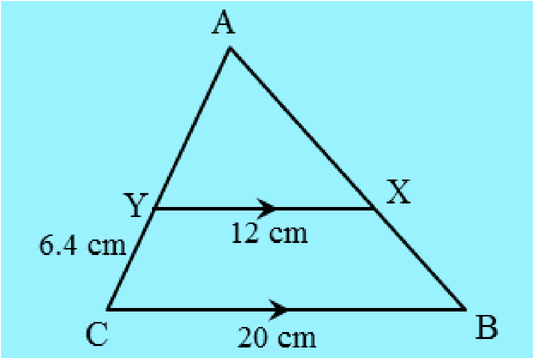
$\therefore$ AC = 17.5 cm $\therefore \frac{AD+4}{AD} = \frac{15}{12} = \frac{5}{4}$

$\therefore$ 5 AD = 4 AD + 16 $\therefore$ AD = 16 cm

$\therefore$ AB = 16 + 4 = 20 cm

Example ❸

In the opposite figure:



BC = 20 cm and CY = 6.4 cm

In Δ ABC, draw $\overline{XY}$ // $\overline{BC}$

If YX = 12 cm, then find the length of $\overline{AC}$

Solution

∵ $\overline{XY}$ // $\overline{BC}$ ∴ Δ ABC ~ Δ AXY

$$\therefore \frac{AB}{AX} = \frac{BC}{XY} = \frac{AC}{AY} \qquad \therefore \frac{AB}{AX} = \frac{20}{12} = \frac{AY + 6.4}{AY}$$

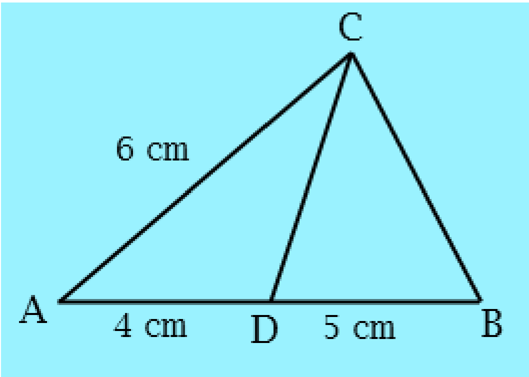
$$\therefore \frac{AY + 6.4}{AY} = \frac{20}{12} = \frac{5}{3}$$

∴ 5 AY = 3 AY + 19.2

∴ 2 AY = 19.2 cm ∴ AY = 9.6 cm ∴ AC = 9.6 + 6.4 = 16 cm

Example ❹

In the opposite figure:



ABC is a triangle in which:

AC = 6 cm, AD = 4 cm and BD = 5 cm

(1) Prove that Δ ACD ~ Δ ABC

(2) $\overline{AC}$ is a tangent to the circumcircle of Δ BCD

Solution

In Δ ACD and Δ ABC:

∵ ∠ A is a common angle

$$\because \frac{AC}{AB} = \frac{6}{9} = \frac{2}{3} \quad \text{and} \quad \frac{AD}{AC} = \frac{4}{6} = \frac{2}{3}$$

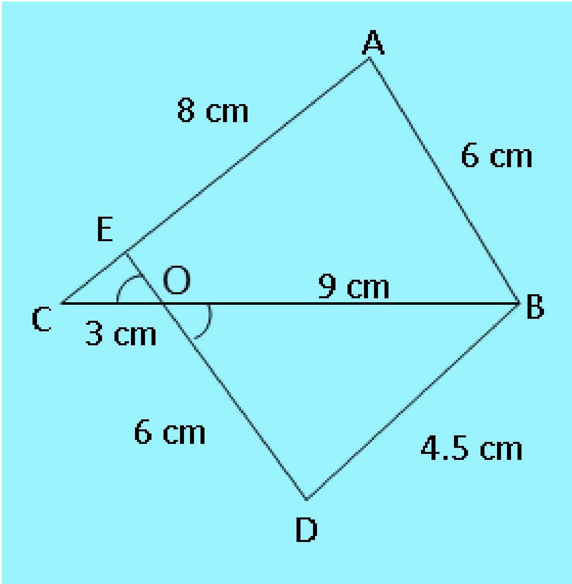
$$\therefore \frac{AC}{AB} = \frac{AD}{AC}$$

∴ Δ ACD ~ Δ ABC ⟶ **(1)**

∴ m (∠ ACD) = m (∠ ABC)

∴ $\overline{AC}$ is a tangent to the circumcircle of Δ BCD ⟶ **(2)**

Example ❺



In the opposite figure:

AB = 6 cm, BC = 12 cm, CA = 8 cm

OC = 3 cm, DB = 4.5 cm, OD = 6 cm.

Prove that:

(1) Δ ABC ~ Δ DBO.

(2) Δ EOC is isosceles.

Solution

In Δ ABC and Δ DBO:

$\because \frac{AB}{DB} = \frac{6}{4.5} = \frac{4}{3}$

$\because \frac{BC}{BO} = \frac{12}{9} = \frac{4}{3}$

$\because \frac{AC}{DO} = \frac{8}{6} = \frac{4}{3}$

$\therefore \frac{AB}{DB} = \frac{BC}{BO} = \frac{AC}{DO} = \frac{4}{3}$

$\therefore$ Δ ABC ~ Δ DBO $\longrightarrow$ **(1)**

$\therefore$ m (∠ BCA) = m (∠ BOD) ***From similarity***

$\because$ m (∠ BOD) = m (∠ EOC) ***V.O.A.***

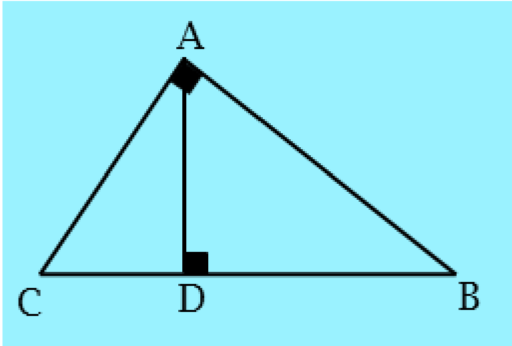
$\therefore$ m (∠ BCA) = m (∠ EOC)

$\therefore$ Δ EOC is isosceles $\longrightarrow$ **(2)**

Corollary

Δ DAB ~ Δ ACB ~ Δ DCA (Why?)

Deduce the Euclid's theorem:



(1) $(AB)^2 = BD \cdot BC$ **(2)** $(AC)^2 = CD \cdot BC$

(3) $(AD)^2 = BD \cdot CD$ **(4)** $AD = \dfrac{AB \times AC}{BC}$

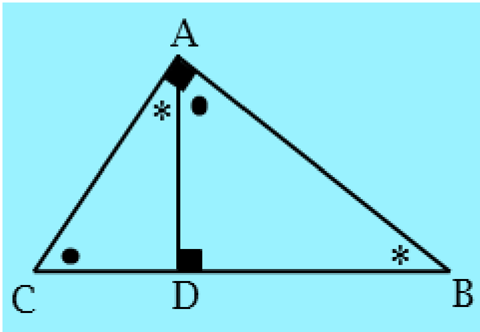
Proof

$\because \Delta DAB \sim \Delta ACB$

$\therefore \dfrac{DA}{AC} = \dfrac{AB}{CB} = \dfrac{DB}{AB}$ $\therefore (AB)^2 = BD \cdot CB$ ①

$\because \Delta DCA \sim \Delta ACB$

$\therefore \dfrac{DC}{AC} = \dfrac{CA}{CB} = \dfrac{DB}{AB}$ $\therefore (AC)^2 = DC \cdot CB$ ②



$\because \Delta DAB \sim \Delta DCA$

$\therefore \dfrac{DA}{DC} = \dfrac{AB}{CA} = \dfrac{DB}{DA}$ $\therefore (AD)^2 = BD \cdot CD$ ③

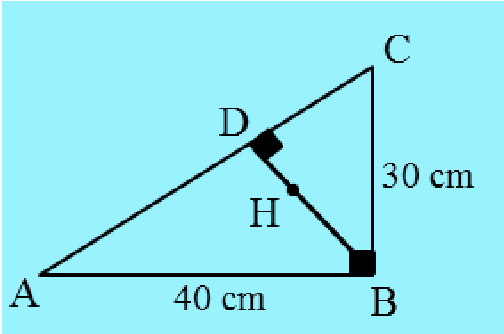
$\because \Delta DAB \sim \Delta ACB$

$\therefore \dfrac{DA}{AC} = \dfrac{AB}{CB} = \dfrac{DB}{AB}$ $\therefore AD = \dfrac{AB \times AC}{BC}$ ④

Example ❻

$\overline{AH}$ and $\overline{BC}$ are two intersecting chords at D in a circle, where D is the midpoint of $\overline{BC}$. Prove that: $(BD)^2 = AD \times DH$

Solution



In Δ ABC:

∵ m (∠ B) = 90°

∴ $AC = \sqrt{(AB)^2 + (BC)^2} = \sqrt{40^2 + 30^2} = 50$ cm

∵ $\overrightarrow{BD} \perp \overline{AC}$

∴ $BD = \frac{AB \times BC}{AC} = \frac{40 \times 30}{50} = 24$ cm

∵ 3 BH = 5 HD ∴ $\frac{BH}{HD} = \frac{5}{3}$ ∴ each part $= \frac{24}{8} = 3$ cm

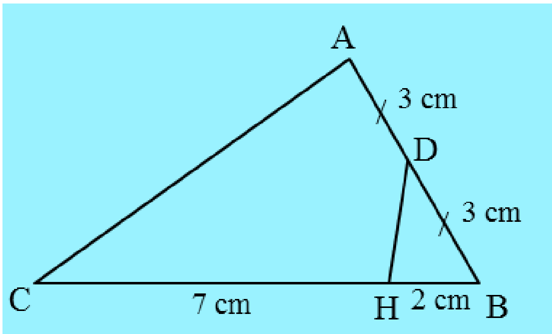
∴ BH = 5 × 3 = 15 cm & ∴ HD = 3 × 3 = 9 cm

Example ❼

If AB = 6 cm and BC = 9 cm. in Δ ABC let point D be the midpoint of $\overline{AB}$ and H ∈ $\overline{BC}$ such that BH = 2 cm.

Prove that: (1) Δ DBH ~ Δ CBA (2) ADHC is a cyclic quadrilateral

Solution



In Δ DBH and Δ CBA:

∵ $\frac{BD}{BC} = \frac{3}{9} = \frac{1}{3}$ ∵ $\frac{BH}{BA} = \frac{2}{6} = \frac{1}{3}$

∴ $\frac{BD}{BC} = \frac{BH}{BA}$ ∵ ∠ B is common angle

∴ Δ DBH ~ Δ CBA ⟶ (1)

∴ m (∠ BDH) = m (∠ BCA)

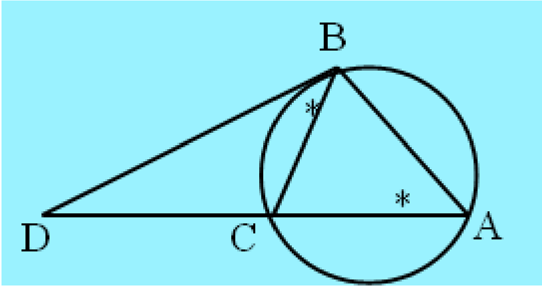
∴ ADHC is a cyclic quadrilateral ⟶ (2)

Example ❽

Δ ABC inscribed in circle, $\overline{BD}$ is a tangent to the circle at B cut $\overrightarrow{AC}$ at D.

Show that: **(1) Δ DBA ~ Δ DCB** **(2) $(DB)^2 = (DA)(DC)$**

Solution



In Δ DBA and Δ DCB:

∵ $\overline{BD}$ is a tangent to the circle at B

∴ m (∠ DAB) = m (∠ DBC) (on $\widehat{BC}$)

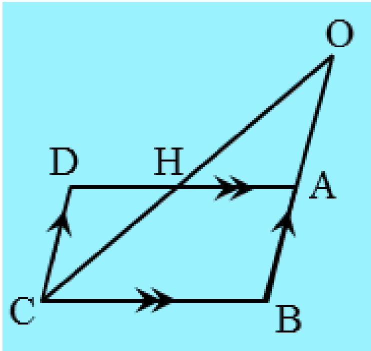
(*the measure of the inscribed angle = the measure of the tangency angle*)

∵ ∠ D is a common angle

∴ m (∠ DBA) = m (∠ DCB) ∴ Δ DBA ~ Δ DCB ⟶ **(1)**

∴ $\frac{DB}{DC} = \frac{BA}{CB} = \frac{DA}{DB}$ ∴ $(DB)^2 = (DA)(DC)$ ⟶ **(2)**

Example ❾



In the opposite figure:

ABCD is a parallelogram.

Prove that: Δ CDH ~ Δ OBC

Solution

∵ ABCD is a parallelogram

∴ $\overrightarrow{OA}$ // $\overline{CD}$

∴ Δ CDH ~ Δ OAH ⟶ ①

∵ $\overline{AH}$ // $\overline{BC}$

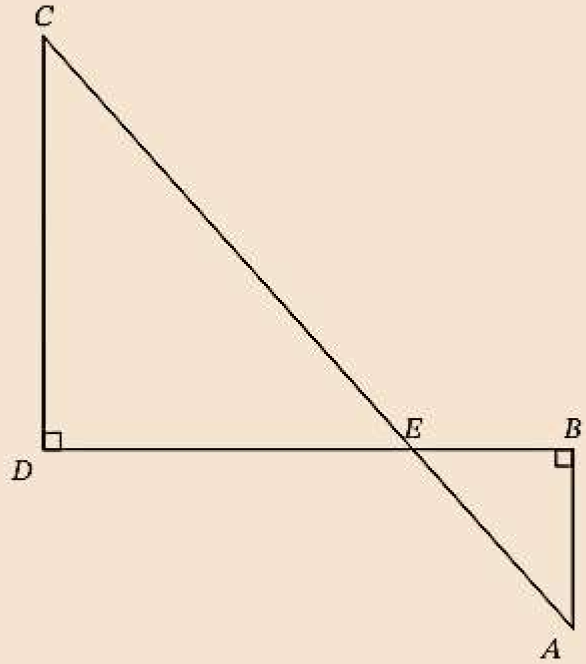
∴ Δ OBC ~ Δ OAH ⟶ ②

From ① & ②: ∴ Δ CDH ~ Δ OBC **[Discus another solution?!]**

PRACTICE

Q1: Given that $DE = 74$ m, $EB = 32$ m,

and $EA = 48$ m, find the length of $\overline{CA}$



Q2: In the given figure, $\overline{DE}$ and $\overline{BC}$ are parallel. Use similarity to work out the value of x.

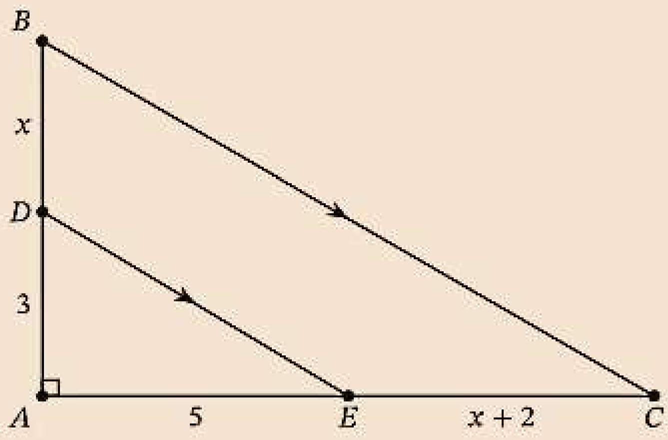
A $x = \frac{6}{8}$

B $x = 3$

C $x = 1$

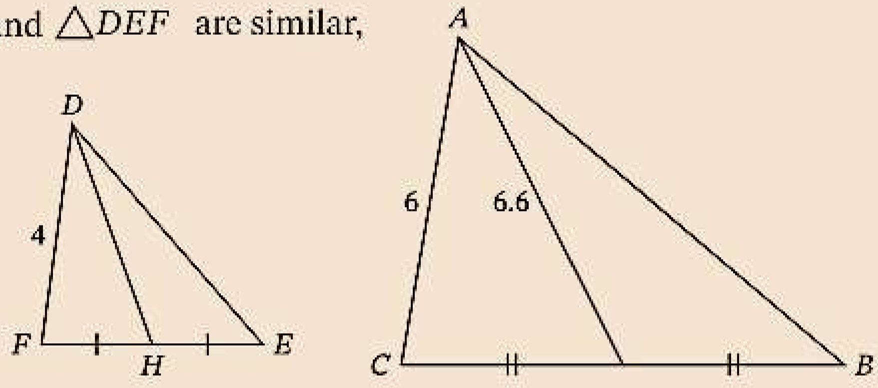
D $x = 6$

E $x = 5$



Q3: Given that $\triangle ABC$ and $\triangle DEF$ are similar,

find the length DH.



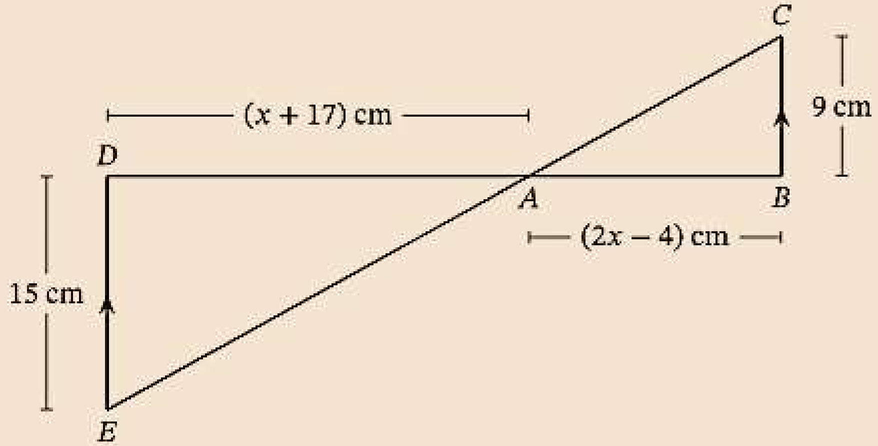
Q4: If $\triangle ABC \sim \triangle ADE$, evaluate x.

A 97

B $\frac{31}{13}$

C $\frac{71}{7}$

D $\frac{73}{11}$



Q5: Given that triangles ABC and $A'B'C'$ are similar, work out the value of x.

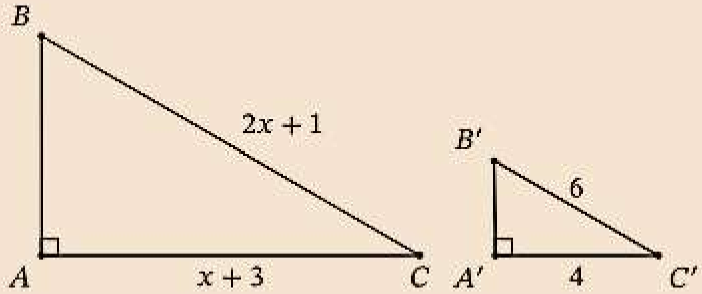
A $x = 12$

B $x = 7$

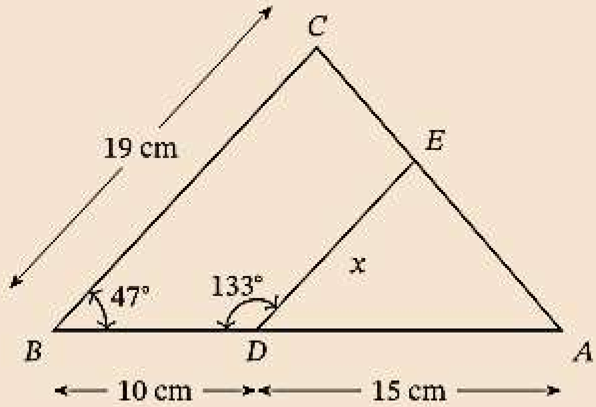
C $x = 5$

D $x = 2.5$

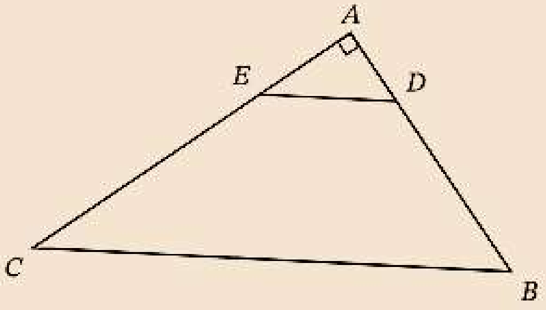
E $x = 6$



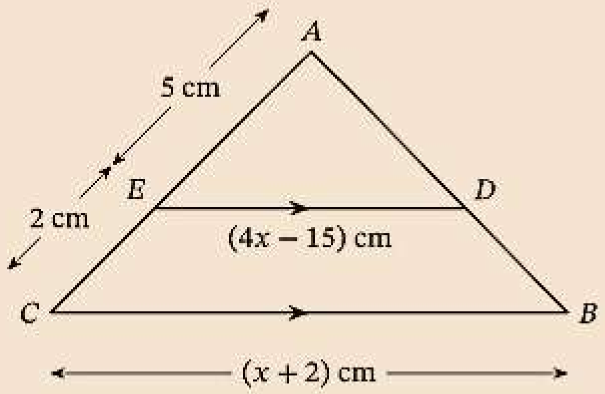
Q6: Find the value of x rounded to the nearest hundredth.



Q7:: Given that $ED = 10$ cm, $AE = 8$ cm, $EC = 20$ cm, and $BD = 15$ cm, find the length of $\overline{BC}$.



Q8: Triangles ABC and ADE are similar. Find x to the nearest integer.



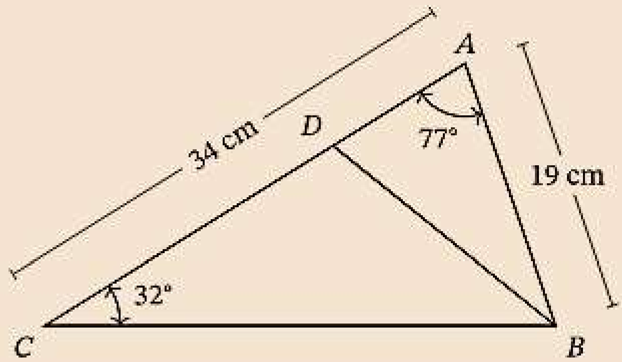
Q9: If $\triangle ABD \sim \triangle ACB$, find $m\angle DBC$ and the length of $\overline{CD}$ to the nearest tenth.

A. 71°, 23.4 cm

B. 32°, 19 cm

C. 39°, 23.4 cm

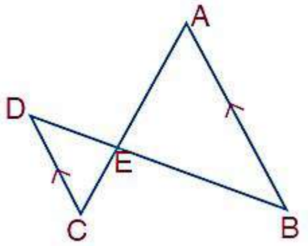
D. 39°, 15 cm



2 - 2 Similarity Of Triangles

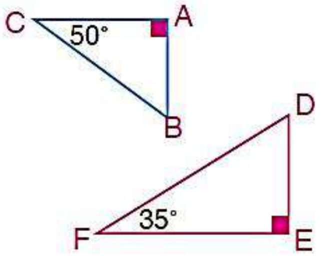
1 State which of the following cases, the two triangles are similar. In case of similarity, state why they are similar?

A



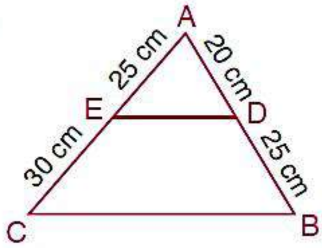
....

B



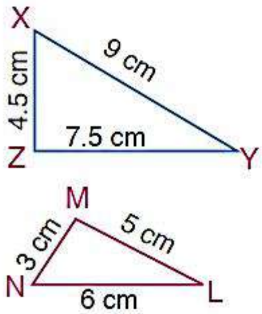
....

C



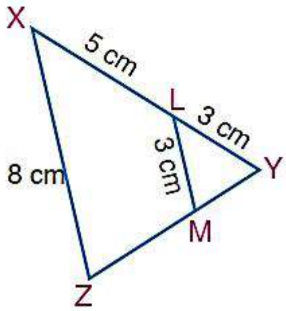
....

D



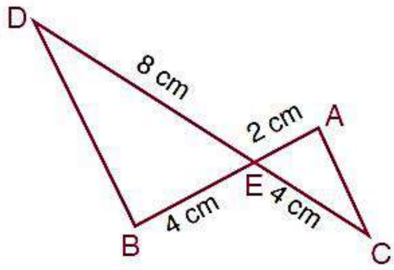
....

E



....

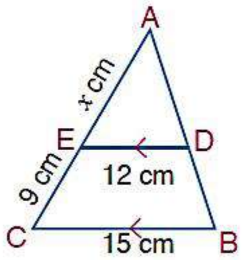
F



....

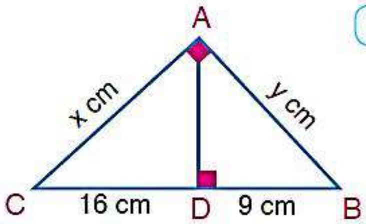
2 Find the value of the symbol used:

A



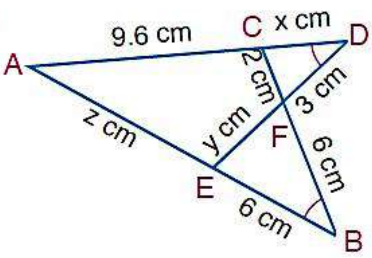
....

B



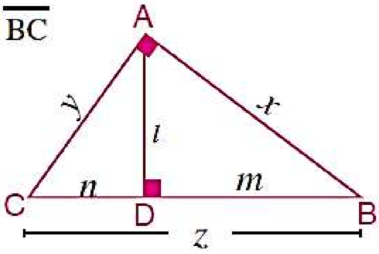
....

C



....

3 **In the figure opposite:** ABC is a right angled triangle, $\overline{AE} \perp \overline{BC}$



First: complete: $\triangle ABC \sim \triangle$ .... $\sim \triangle$ ....

Second: If x, y, z, l, m and n are the lengths of the straight segments in centimetres, then complete the following proportions:

A $\frac{x}{z} = \frac{m}{\cdots\cdots}$ **B** $\frac{x}{z} = \frac{l}{\cdots\cdots}$ **C** $\frac{m}{l} = \frac{x}{\cdots\cdots}$ **D** $\frac{l}{\cdots\cdots} = \frac{\cdots\cdots}{l}$

E $\frac{x}{\cdots\cdots} = \frac{\cdots\cdots}{x}$ **F** $\frac{\cdots\cdots}{y} = \frac{y}{\cdots\cdots}$ **G** $\frac{l}{x} = \frac{\cdots\cdots}{z}$ **H** $\frac{l}{x} = \frac{\cdots\cdots}{y}$

4 $\overline{AB}$ and $\overline{DC}$ are two chords in a circle, $\overrightarrow{AB} \cap \overrightarrow{DC} = \{E\}$, where E lies outside the circle, AB = 4cm, DC = 7cm and BE = 6cm. prove that $\triangle ADE \sim \triangle CBE$, then find the length of $\overline{CE}$

....

5 ABC, and DEF are two similar triangles, $\overrightarrow{AX} \perp \overline{BC}$ to intersect it at X, $\overrightarrow{DY} \perp \overline{EF}$ and intersects it at Y. Prove that $BX \times YF = CX \times YE$

....

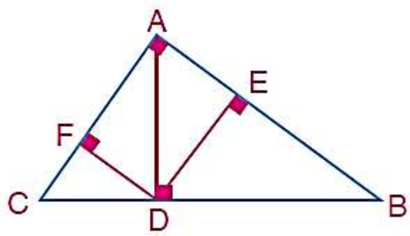
6 In $\triangle ABC$, $AC > AB$, $M \in \overline{AC}$ where $m(\angle ABM) = m(\angle C)$.
Prove that $(AB)^2 = AM \times AC$.

....

7 ABC is a right angled triangle at A, $\overrightarrow{AD} \perp \overline{BC}$ to intersect it at D . if $\frac{BD}{DC} = \frac{1}{2}$, $AD = 6\sqrt{2}$ cm. Find the length of $\overline{BD}$, $\overline{AB}$ and $\overline{AC}$.

....

8 **In the figure opposite:** ABC is a right angled triangle at A,



$\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{AC}$. Prove that:

A $\triangle ADE \sim \triangle CDF$

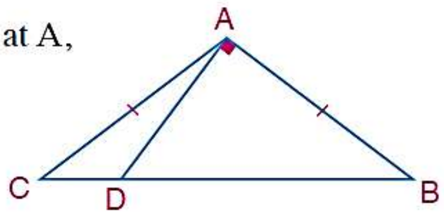
B Area of rectangle AEDF $= \sqrt{AE \times EB \times AF \times FC}$

....

9 **In the figure opposite:** ABC is an obtuse angled triangle at A,

AB = AC. $\overrightarrow{AD} \perp \overline{AB}$ and intersects $\overline{BC}$ at D.

Prove that: $2(AB)^2 = BE \times BC$



....

....

10 The two sets A and B represent the side lengths of different triangles in centimetres. In front of each triangle from set A Write the triangle similar to it from set B

Set (A)

1	6	6	6
2	5	7	11
3	5	8	10
4	7	8	12
5	16	27	28

Set (B)

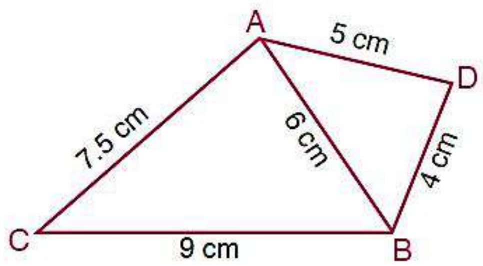
A	2.5	4	5
B	8	13,5	14
C	25	35	55
D	11	11	11
E	3,5	4	6
F	8	6	10
G	32	54	42

11 **In the figure opposite:** A B C is a triangle in which AB = 6cm ,

BC = 9cm and AC = 7,5cm.

D is a point outside the triangle ABC

where D B = 4cm and DE = 5cm. Prove that:



A $\triangle ABC \sim \triangle DBA$

B $\overrightarrow{BA}$ bisects $\angle DBC$

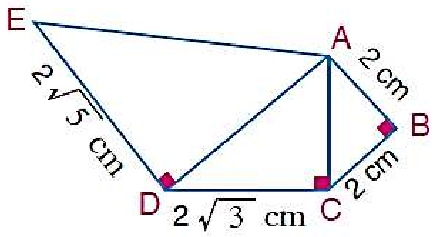
12 **In the figure opposite , Complete:**

$\triangle ABC \sim \triangle$ ....

and the scale factor = ...

....

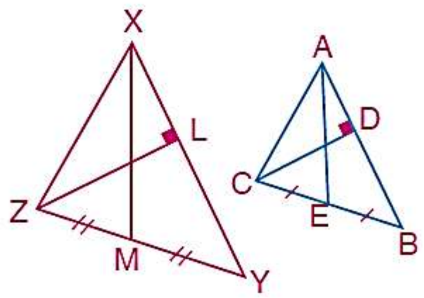
....



(13) **In the figure opposite:** $\triangle ABC \sim \triangle XYZ$, ,

E is the mid point of $\overline{BC}$, M is the mid point of $\overline{YZ}$,

$\overline{CD} \perp \overline{AB}$ and $\overline{ZL} \perp \overline{XY}$. prove that:



(A) $\triangle AEC \sim \triangle XMZ$

(B) $\dfrac{CD}{ZL} = \dfrac{AE}{XM}$

....

....

(14) ABC and XYZ are two similar triangles, where, $AB > AC$, $XY > XZ$.
E and L are the mid point of $\overline{BC}$ and $\overline{YZ}$ respectively. $\overline{AF} \perp \overline{BC}$ and $\overline{XM} \perp \overline{YZ}$
Prove that $\triangle AEF \sim \triangle XLM$

....

(15) ABC is a triangle, $D \in \overline{BC}$ where $(AD)^2 = BD \times DC$, $BA \times AD = BD \times AC$. Prove that:

(A) $\triangle ABD \sim \triangle CAD$ (B) $\overline{AD} \perp \overline{BC}$ (C) $m(\angle BAC) = 90°$

Lesson (3)

The Relation Between the Areas of two Similar Polygons

Lesson objectives

Related Links

- Determine the relation between the perimeters of two similar polygons and similarity ratio (scale factor).
- Recognize the relation between areas of two similar polygons and similarity ratio.

❶ The ratio between the areas of two similar triangles equals the square of the ratio between the lengths of any two corresponding sides of the two triangles.

If Δ ABC ~ Δ DEF $\longrightarrow \frac{A\ (\Delta ABC)}{A\ (\Delta DEF)} = \left(\frac{AB}{DE}\right)^2 = \left(\frac{BC}{EF}\right)^2 = \left(\frac{AC}{DF}\right)^2$

❷ The ratio between the areas of two similar polygons equals the square of the ratio between the lengths of any two corresponding of the two polygons.

If P_1 (ABCDE) ~ P_2 (XYZLO) $\longrightarrow \frac{A\ (P_1)}{A\ (P_2)} = \left(\frac{AB}{XY}\right)^2 = \left(\frac{BC}{YZ}\right)^2 = \left(\frac{CD}{YZ}\right)^2 = \ldots..$

Corollary: The ratio between the perimeters of two similar polygons is equal to the ratio between any two corresponding sides of the two polygons.

Example ❶

Δ ABC ~ Δ DEF & S.A. of (Δ ABC) = 9 S.A of (DEF) & DE = 5 cm. find AB

Solution

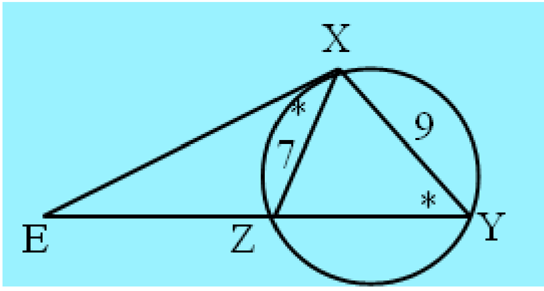
$\because \Delta ABC \sim \Delta DEF$ $\quad \therefore \frac{S.A.\ of\ (\Delta ABC)}{S.A.\ of\ (\Delta DEF)} = \left(\frac{AB}{DE}\right)^2$

$\therefore \frac{9}{1} = \frac{(AB)^2}{5^2}$ $\quad \therefore (AB)^2 = 225$ $\quad \therefore AB = \sqrt{225} = 15$ cm

Example ❷

Δ XYZ in which $\frac{XY}{XZ} = \frac{9}{7}$ the circle passing through the vertices of this Δ is drawn. The tangent at X is drawn to cut $\overrightarrow{YZ}$ at E. prove that: $\frac{A\,(\Delta\ XYZ)}{A\,(\Delta\ XYE)} = \frac{32}{81}$

Solution



In Δ EXY and Δ EZX:

$\because$ $\overline{EX}$ is a tangent to the circle at X

$\therefore$ m ($\angle$ EYX) = m ($\angle$ EXZ) (on $\widehat{XZ}$)

(*the measure of the inscribed angle = the measure of the tangency angle*)

$\because$ $\angle$ E is a common angle

$\therefore$ m ($\angle$ EXY) = m ($\angle$ EZX)

$\therefore$ Δ EXY ~ Δ EZX

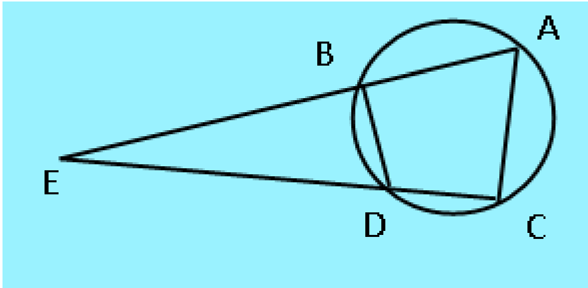
$$\therefore \frac{A\,(\Delta\ EXY)}{A\,(\Delta\ EZX)} = \left(\frac{XY}{ZX}\right)^2 = \left(\frac{9}{7}\right)^2 = \frac{81}{49}$$

$$\therefore \frac{A\,(\Delta\ XYZ)}{A\,(\Delta\ EZX)} = \frac{81-49}{49} = \frac{32}{49}$$

$$\therefore \frac{A\,(\Delta\ XYZ)}{A\,(\Delta\ XYE)} = \frac{32}{81}$$

Example ❸

In the opposite figure:

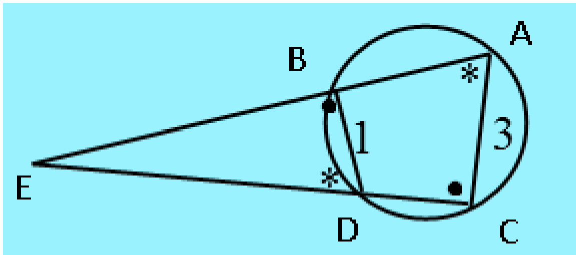


$\overline{AB}$ and $\overline{CD}$ are two non-interesting chords in a circle.

If $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{E\}$ and AC = 3 BD.

Find: $\dfrac{\text{area of the quadrilateral ABDC}}{\text{area of triangle EAC}}$

Solution



$\because$ AC = 3 BD

$\therefore \dfrac{AC}{DB} = \dfrac{3}{1}$

$\because$ ABDC is a cyclic quadrilateral

$\therefore$ m (∠ EAC) = m (∠ EDB)

$\therefore$ m (∠ ECA) = m (∠ EBD)

$\because$ ∠ E is common angle

$\therefore$ Δ EAC ~ Δ EDB

$\therefore \dfrac{A\,(\Delta\ EAC)}{A\,(\Delta\ EDB)} = \left(\dfrac{AC}{DB}\right)^2 = \left(\dfrac{3}{1}\right)^2 = \dfrac{9}{1}$

$\therefore \dfrac{A\,(\text{quad. }ABDC)}{A\,(\Delta\ EDB)} = \dfrac{9-1}{1} = \dfrac{8}{1}$

$\therefore \dfrac{\text{area of the quadrilateral ABDC}}{\text{area of triangle EAC}} = \dfrac{8}{9}$

Example ❹

① **The ratio between the lengths of two corresponding sides of two similar polygon is 1 : 2 what is the ratio between there areas and what is the ratio between their perimeters.**

Solution

$\because \frac{L_1}{L_2} = \frac{1}{2}$	$\therefore \frac{A_1}{A_2} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\therefore \frac{P_1}{P_2} = \frac{L_1}{L_2} = \frac{1}{2}$

② **The ratio between the areas of two similar polygon is 4 : 9 what is the ratio between their corresponding sides, what is the ratio between their perimeters.**

Solution

$\because \frac{A_1}{A_2} = \frac{4}{9}$	$\therefore \frac{L_1}{L_2} = \sqrt{\frac{4}{9}} = \frac{2}{3}$	$\therefore \frac{P_1}{P_2} = \frac{L_1}{L_2} = \frac{2}{3}$

③ **The ratio between the perimeters of two similar polygons is 3 : 4 if the area of first polygon is 45 cm², Find area of the second polygon.**

Solution

$\because \frac{P_1}{P_2} = \frac{3}{4}$	$\therefore \frac{A_1}{A_2} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$	$\therefore \frac{45}{P_2} = \frac{9}{16}$	$\therefore P_2 = 80\ cm^2$

④ **The ratio is between the lengths of two corresponding sides of two similar polygons is 2 : 3, if the sum of areas of the two a polygons equals 143 cm², Find the area of each.**

Solution

$\because \frac{L_1}{L_2} = \frac{2}{3}$

$\therefore \frac{A_1}{A_2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

A_1	:	A_2	:	sum
4	:	9	:	13
	:		:	143

$\therefore A_1 = 44\ cm^2$

And

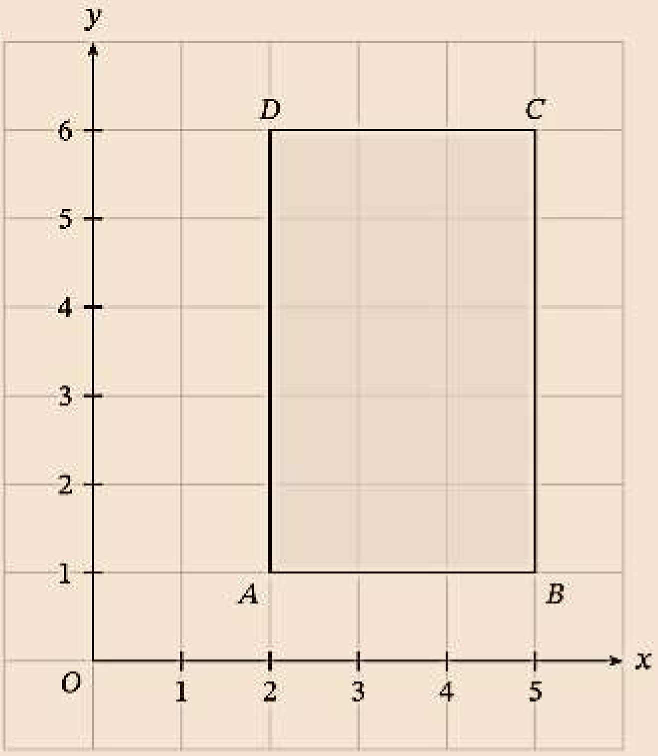
$\therefore A_1 = 99\ cm^2$

PRACTICE

Q1: Given the following figure,

find the area of a similar polygon

$A'B'C'D'$ in which $A'B' = 6$.

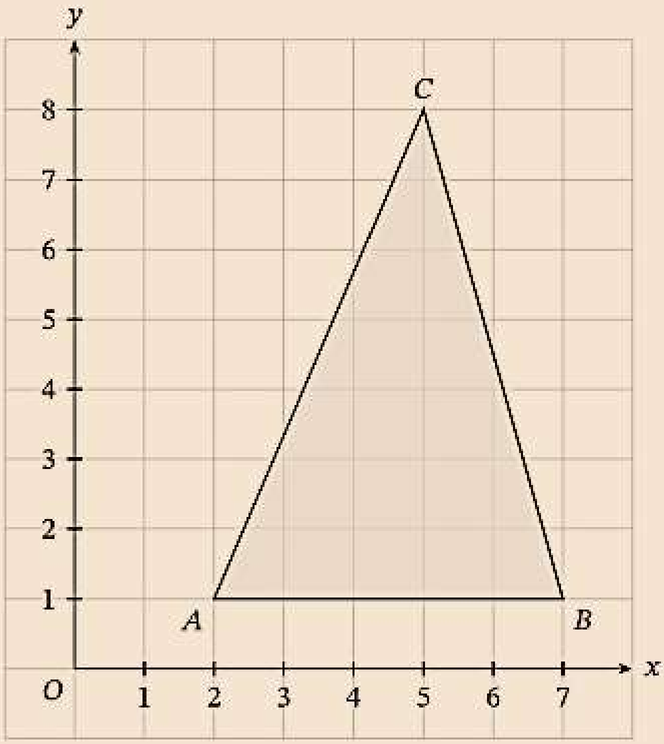


Q2: Given the figure shown,

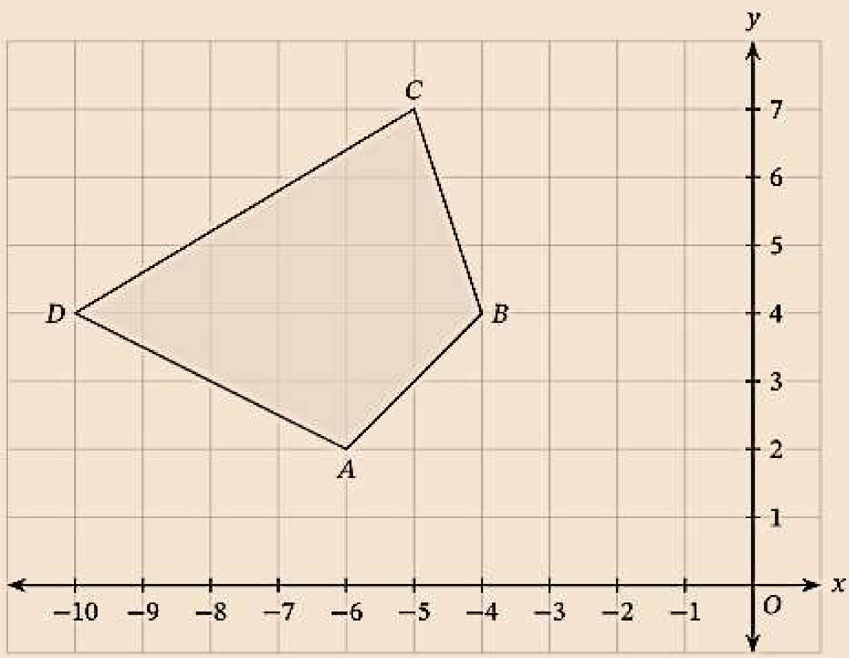
determine the area of a

similar polygon, $A'B'C'$

, in which $A'B' = 3$.



Q3: Given the graph, determine the area of similar polygon $A'B'C'D'$ in which $B'C' = 6$.



Q4: Rectangle $QRST$ is similar to rectangle $JKLM$ with their sides having a ratio of $8 : 9$. If the dimensions of each rectangle are doubled, find the ratio of the areas of the larger rectangles.

A $4 : 9$

B $16 : 9$

C $32 : 81$

D $64 : 81$

E $128 : 81$

Q5: Two similar polygons have areas of 20 in^2 and 80 in^2. Find the scale factor of the first polygon to the second.

A $1 : 5$

B $1 : 2$

C $1 : 4$

D $4 : 1$

E $2 : 1$

Q6: Two corresponding sides of two similar polygons have lengths of 54 and 57 centimetres. Given that the area of the smaller polygon is 324 cm^2, determine the area of the bigger polygon.

Q7: $ABCD$ is a square where AB, BC, CD, and DA are divided by the points X, Y, Z, and L, respectively, by the ratio of $4 : 1$. Find the ratio of the area of $XYZL$ to that of $ABCD$.

A. $3 : 5$

B. $25 : 17$

C. $489 : 593$

D. $17 : 25$

E. $593 : 489$

Q8: If $\triangle ABC \sim \triangle XYZ$ and $AB = \frac{9}{5}XY$, find $\dfrac{\text{area of } XYZ}{\text{area of } ABC}$.

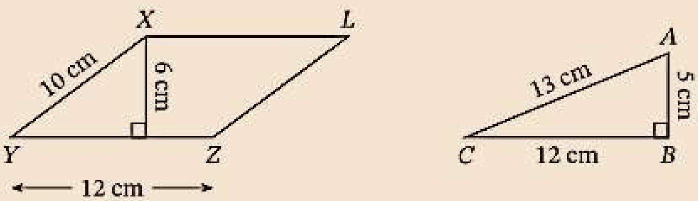
A. $\frac{36}{5}$

B. $\frac{9}{5}$

C. $\frac{18}{5}$

D. $\frac{25}{81}$

Q9: Using the figure below, find the ratio between the area of the parallelogram $XYZL$ and the area of the triangle ABC in its simplest form.



A. $5 : 3$

B. $12 : 5$

C. $5 : 6$

D. $22 : 15$

E. $2 : 1$

Q10: Triangle ABC is right angled at A, where $AB = 20$ and $AC = 21$. Suppose L, M, and N are similar polygons on corresponding sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. If the area of L is 145, what are the areas of M and N to the nearest hundredth?

A area of $M = 159.86$, area of $N = 304.86$

B area of $M = 304.86$, area of $N = 159.86$

C area of $M = 210.25$, area of $N = 152.25$

D area of $M = 17.46$, area of $N = 12.64$

Q10: Rectangle $QRST$ is similar to rectangle $JKLM$ with their sides having a ratio of $5 : 3$. If the dimensions of each rectangle are tripled, find the ratio of the areas of the larger rectangles.

A $5 : 9$

B $5 : 1$

C $25 : 27$

D $25 : 9$

E $25 : 3$

Q11: Rectangle $QRST$ is similar to rectangle $JKLM$ with their sides having a ratio of $9 : 5$. If the dimensions of each rectangle are tripled, find the ratio of the areas of the larger rectangles.

A $3 : 5$

B $27 : 5$

C $27 : 25$

D $81 : 25$

E $243 : 25$

Q12: Rectangle $QRST$ is similar to rectangle $JKLM$ with their sides having a ratio of $4 : 7$. If the dimensions of each rectangle are doubled, find the ratio of the areas of the larger rectangles.

A $2 : 7$

B $8 : 7$

C $8 : 49$

D $16 : 49$

E $32 : 49$

2 - 3 The Relation between the Areas of two Similar Polygons

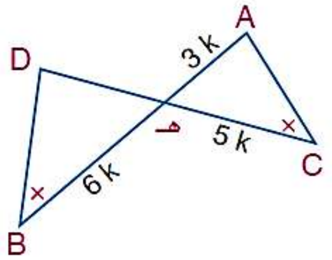
1 Complete:

A If $\triangle ABC \sim \triangle XYZ$, and $AB = 3\,XY$, then $\dfrac{\text{area }(\triangle XYZ)}{\text{area }(\triangle ABC)} =$ ....

B If $\triangle ABC \sim \triangle DEF$, area of $(\triangle ABC) = 9$ area of $(\triangle DEF)$ and $DE = 4$cm, then $AB =$ .... cm

2 Study each of the following figures, where K is constant of proportion , then complete:

A

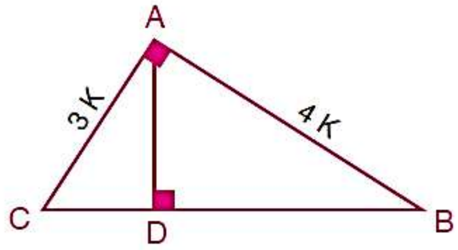


$\overline{AB} \cap \overline{CD} = \{E\}$

area of $(\triangle ACE) = 900\ cm^2$

then: area of $(\triangle DEB) =$ .... cm^2

B



$m(\angle BAC) = 90°$, $\overline{AD} \perp \overline{BC}$

area of $(\triangle ADC) = 180\ cm^2$ then:

area of $(\triangle ABC) =$ .... cm^2

3 ABC is a triangle, $D \in \overline{AB}$ where $AD = 2\,BD$ and $E \in \overline{AC}$ where $\overline{DE} \parallel \overline{BC}$

If the area of $\triangle ADE = 60cm^2$, find the area of the trapezium DBCE.

....

....

4 ABC is a right angled triangle at B. The equilateral triangles ABX, BCY, and ACZ are drawn . prove that : area of $(\triangle ABX)$ + area of $(\triangle BCY)$ = area of $(\triangle ACZ)$.

....

....

5. ABC is an inscribed triangle in a circle where $\frac{AB}{BC} = \frac{4}{3}$. from the point B, a tangent is drawn to the circle and intersects $\overrightarrow{AC}$ at E.

prove that: $\frac{\text{area of }(\triangle ABC)}{\text{area of }(\triangle ABE)} = \frac{7}{16}$

....

....

6. ABCD is a parallelogram, $X \in \overrightarrow{AB}$, $X \notin \overline{AB}$, where $BX = 2AB$, $Y \in \overrightarrow{CB}$, $Y \notin \overline{CB}$,

where $BY = 2BC$. the parallelogram BXZY is drawn, prove that: $\frac{\text{area of }(ABCD)}{\text{area of }(XBYZ)} = \frac{1}{4}$

....

....

7. A B C is a right angled triangle at B, $\overline{BD} \perp \overline{AC}$ and intersects it at D. The squares AXYB and BMNC are drawn on $\overline{AB}$ and $\overline{BC}$ respectively outside the triangle ABC:

A. Prove that polygon DAXY B = polygon DBMNC

B. If AB = 6cm and AC = 10cm, find the ratio between the areas of the two polygons.

....

....

8. A B C is a triangle, $\overline{AB}$, $\overline{CB}$ and $\overline{AC}$ are corresponding sides to three similar polygons X, Y, Z drawn outside the triangle respectively. If the area of polygon X equals 40 cm^2, area of polygon Y equals 85 cm^2 and area of polygon Z equals 125cm^2.
Prove that: the triangle ABC is right angled.

....

....

9. ABCD is a square, $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ are divided in the ratio 1 : 3 by the points X, Y, Z and L respectively.

Prove that:

A. XYZL is a square

B. $\frac{\text{Area of the square XYZL}}{\text{Area of the square ABCD}} = \frac{5}{8}$

....

Lesson (4) Applications of Similarity in the circle

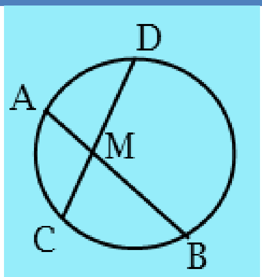
Lesson objectives

Related Links

- Identify the relation between two intersecting chords in a circle.
- Classify the relation between two secants to the circle from a point outside it.
- Solve problems, and life applications using similarity of polygons in a circle.

Well known Problems

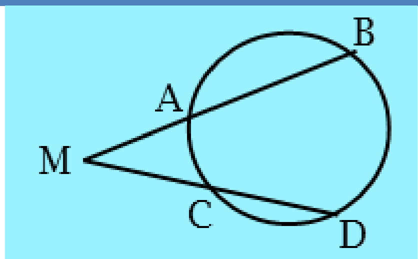
①



If $\overrightarrow{BA} \cap \overrightarrow{DC} = \{M\}$

$\therefore MA \times MB = MC \times MD$

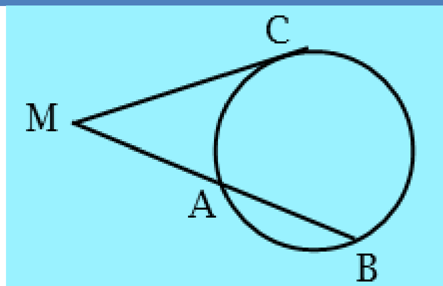
②



If $\overline{AB} \cap \overline{CD} = \{M\}$

$\therefore MA \times MB = MC \times MD$

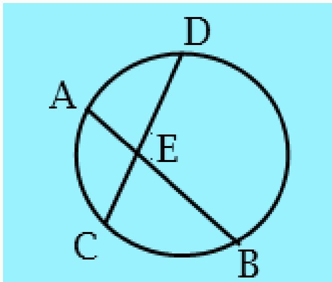
③



If $\overline{MC}$ is a tangent

$\therefore (MC)^2 = MA \times MB$

Example ❶



❶ **Complete:**

(1) EA . EB = ……………………..

Solution: $EA \times EB = EC \times ED$

(2) If: AE = 30 cm, EC = 10 cm. and ED = 12 cm, then EB = ……………………..

Solution: $\because EA \times EB = EC \times ED$ $\quad \therefore 30 \times EB = 10 \times 12$ $\quad \therefore EB = 4$ cm

(3) If: EA = 3 cm, EB = 6 cm. and EC = 2 cm, then ED = ……………………………

Solution: $\because EA \times EB = EC \times ED$ $\quad \therefore 3 \times 6 = 10 \times ED$ $\quad \therefore ED = 1.8$ cm

(4) If: EA = $2x$ cm, EB = $3x$ cm., EC = 3 cm and ED = 8 cm, then x = ……………

Solution: $\because EA \times EB = EC \times ED$ $\quad \therefore 2x \times 3x = 3 \times 8$

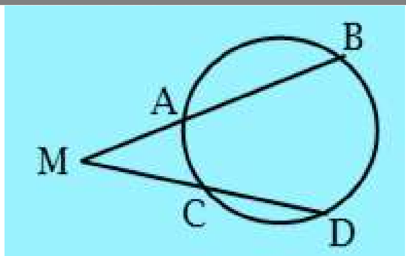
$\therefore 6x^2 = 24$ $\quad \therefore x^2 = 4$ $\quad \therefore x = \sqrt{4} = 2$ cm

(5) If: EA = $(15 - x)$ cm, EB = x cm., EC = 9 cm and ED = 4 cm, then x = ……..

Solution: $\because EA \times EB = EC \times ED$ $\quad \therefore (15 - x) \times x = 9 \times 4$

$\therefore 15x - x^2 = 36$ $\quad \therefore x^2 - 15x + 36 = 0$ $\quad \therefore x = 12$ or $x = 3$

❷ **Complete:**



(1) MA × MB = ...

Solution: MA × MB = MC × MD

(2) MA = 3 cm, AB = x cm, MC = 2 cm and CD = 6.5 cm, then x = ...

Solution: $\because$ MA × MB = MC × MD $\quad \therefore 3 \times (3 + x) = 2 \times 8.5$

$\therefore 9 + 3x^2 = 18 \quad \therefore 3x^2 = 9 \quad \therefore x^2 = 3 \quad \therefore x = \sqrt{3}$ cm

(3) MA = x cm, AB = 2x cm, MC = 3 cm and CD = 6 cm, then x = ...

Solution: $\because$ MA × MB = MC × MD $\quad \therefore x \times (x + 2x) = 3 \times 9$

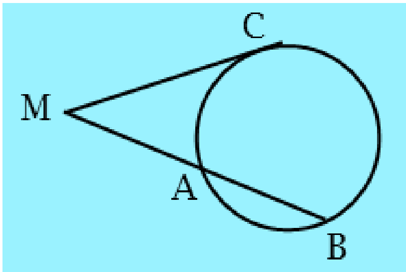
$\therefore x \times 3x = 27 \quad \therefore 3x^2 = 27 \quad \therefore x^2 = 9 \quad \therefore x = 3$ cm

(4) MA = x cm, AB = 5 cm, MC = 3 cm and CD = 9 cm, then x = ...

Solution: $\because$ MA × MB = MC × MD $\quad \therefore x \times (x + 5) = 3 \times 12$

$\therefore x^2 + 5x = 36 \quad \therefore x^2 + 5x - 36 = 0 \quad \therefore x = 4$ cm

❸ **Complete:**



(1) MA. MB = ...

Solution: MA × MB = $(MC)^2$

(2) MC = x cm, MA = 4 cm and AB = 5 cm, then x = ...

Solution: $\because$ MA × MB = $(MC)^2$ $\quad \therefore 4 \times 9 = x^2$

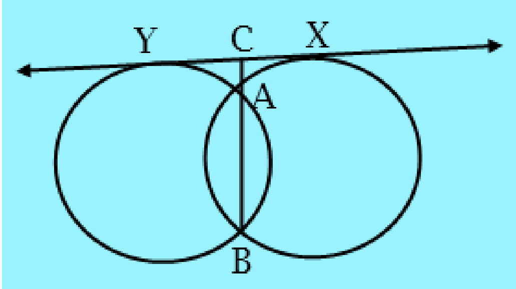
$\therefore x^2 = 36 \quad \therefore x = \sqrt{36} \quad \therefore x = 6$ cm

(3) MC = 8 cm, MA = x cm and AB = 12 cm, then x = ...

Solution: $\because$ MA × MB = $(MC)^2$ $\quad \therefore x \times (x + 12) = 8^2$

$\therefore x^2 + 12x = 64 \quad \therefore x^2 + 12x - 64 = 0 \quad \therefore x = 4$ cm

Example ❷



In the opposite figure:

A and B are the points of intersection of two circles. A common tangent is drawn touching them at X & Y.

If $\overleftrightarrow{AB} \cap \overleftrightarrow{XY} = \{C\}$ show that C is mid-point of $\overline{XY}$.

Solution

$\because \overrightarrow{CX}$ is a tangent at X $\quad \therefore (CX)^2 = CA \times CB \quad$ **(1)**

$\because \overrightarrow{CY}$ is a tangent at Y $\quad \therefore (CY)^2 = CA \times CB \quad$ **(2)**

From **(1)** and **(2)**: $\quad \therefore (CX)^2 = (CY)^2$

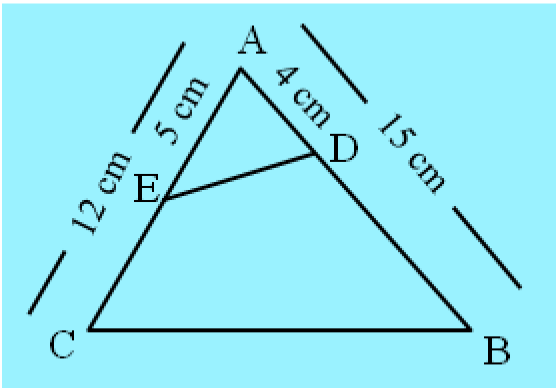
$\therefore CX = CY \quad \therefore$ C is mid-point of $\overline{XY}$

Example ❸

ABC is a triangle in which AB = 15 cm, AC = 12 cm. D ∈ $\overline{AB}$ where AD = 4cm, E ∈ $\overline{AC}$ where AE = 5cm.

Prove that the figure DBCE is a cyclic quadrilateral.

Solution



In Δ ABC and Δ AED:

$\because \angle A$ is common angle

$\because \frac{AB}{AE} = \frac{15}{5} = 3 \qquad \because \frac{AC}{AD} = \frac{12}{4} = 3$

$\therefore \frac{AB}{AE} = \frac{AC}{AD}$

$\therefore \Delta ABC \sim \Delta AED$

$\therefore m(\angle ABC) = m(\angle AED)$ $\qquad$ (*the exterior = the interior opposite*)

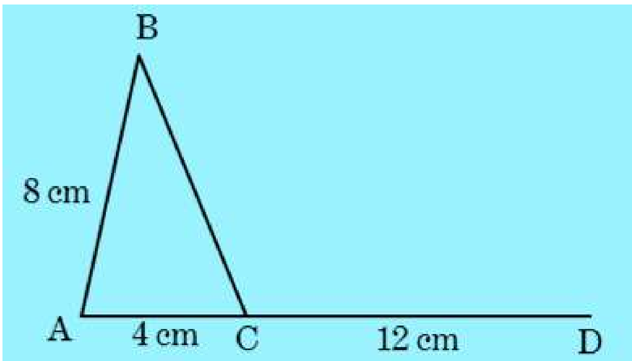
$\therefore$ DBCE is a cyclic quadrilateral

Example ❹

Δ ABC in which AB = 8 cm, AC = 4 cm, D ∈ $\overrightarrow{AC}$, D ∉ $\overline{AC}$ where CD = 12 cm.

Prove that $\overline{AB}$ touches the circle which passes through the points B, C, and D

Solution



∵ $(AB)^2 = 8^2 = 64$

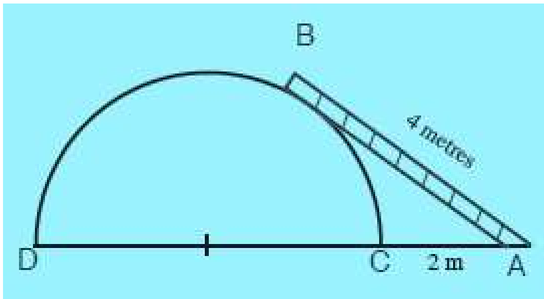
∵ AC × AD = 4 × 16 = 64

∴ $(AB)^2$ = AC × AD

∴ $\overline{AB}$ touches the circle which passes through the points B, C, and D

Example ❺

A ladder of length 4 meters rests on a horizontal rough ground, and with the other end on a hemispheric tank, as in the figure opposite, If the lower end of the ladder is 2 meters far from the base of the tank.



Find the length of the radius of the sphere's tank.

Solution

∵ $(AB)^2$ = AC × AD

∴ 4^2 = 2 × AD

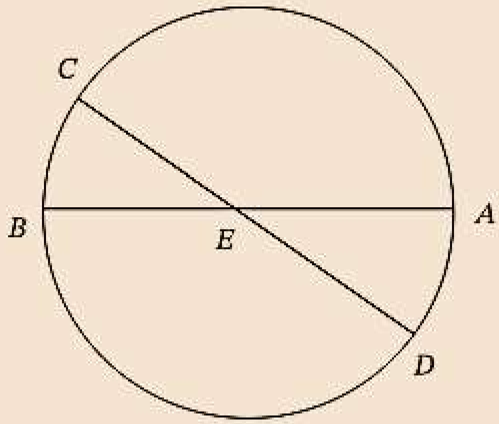
∴ AD = 8 m

∴ CD = 8 – 2 = 6 m

∴ The length of the radius of the sphere's tank = 3 m

PRACTICE

Q1: If $\frac{EA}{EB} = \frac{8}{7}$, $EC = 7$ cm, and $ED = 8$ cm, find the lengths of $\overline{EB}$ and $\overline{EA}$.



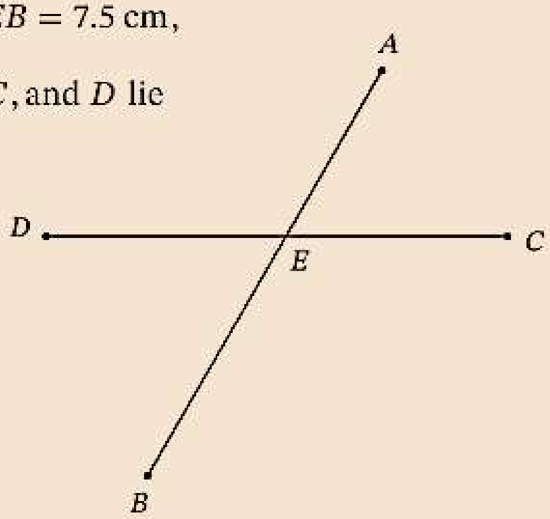
A $EB = 7$ cm, $EA = 8$ cm

B $EB = 6.12$ cm, $EA = 9.14$ cm

C $EB = 8$ cm, $EA = 7$ cm

D $EB = 9.14$ cm, $EA = 6.12$ cm

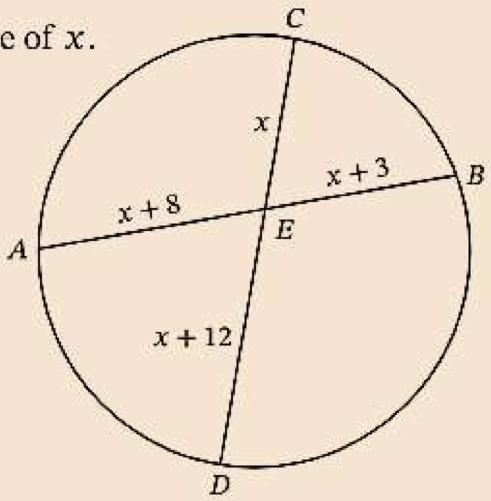
Q2: Given that $EA = 5.2$ cm, $EC = 6$ cm, $EB = 7.5$ cm, and $ED = 6.5$ cm, do the points A, B, C, and D lie on a circle?



A yes

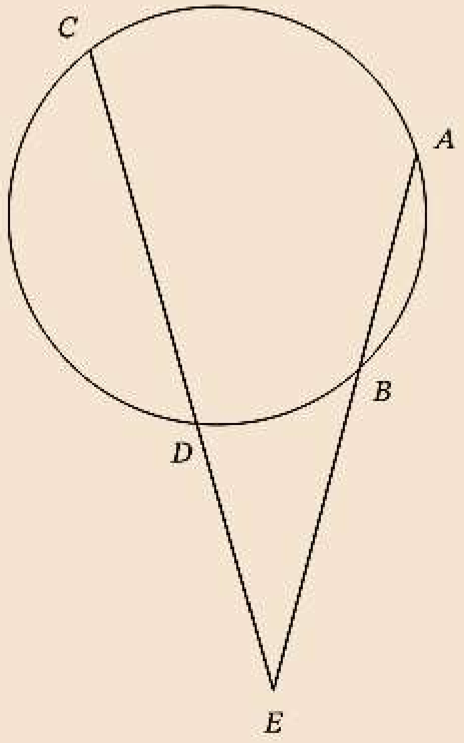
B no

Q3: In the following figure, find the value of x.



Q4: If $\frac{EA}{EB} = \frac{5}{3}$, $EC = 12$ cm, and $ED = 5$ cm

, find the lengths of $\overline{EB}$ and $\overline{BA}$.



A $EB = 10$ cm, $BA = 6$ cm

B $EB = 6$ cm, $BA = 10$ cm

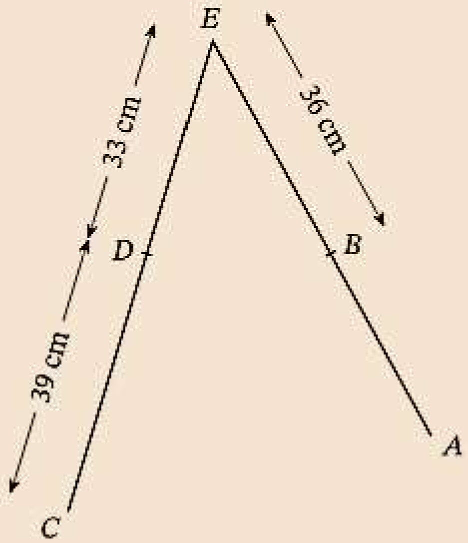
C $EB = 24$ cm, $BA = 20$ cm

D $EB = 6$ cm, $BA = 4$ cm

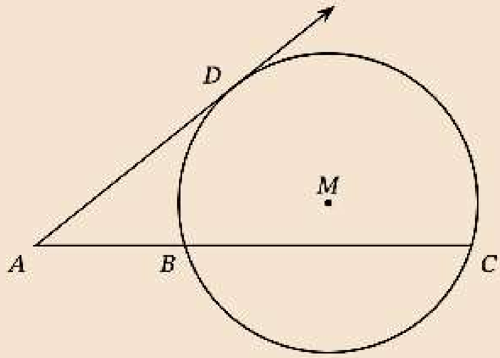
E $EB = 4$ cm, $BA = 6$ cm

Q5: Given that the points A, B, C, and D

lie on a circle, find the length of $\overline{BA}$.



Q6: In the figure shown, the circle has a radius of 12 cm, $AB = 12$ cm, and $AC = 35$ cm. Determine the distance from $\overline{BC}$ to the centre of the circle, M, and the length of $\overline{AD}$, rounding your answers to the nearest tenth.



A 11.5 cm, 20.2 cm

B 3.4 cm, 26.6 cm

C 19.6 cm, 20.2 cm

D 3.4 cm, 20.5 cm

Q7: $\overline{AB}$ and $\overline{CD}$ are two chords of a circle intersecting at H. Find HD, where $AH = 10$, $HB = 8$, and $CH = 16$.

A 2

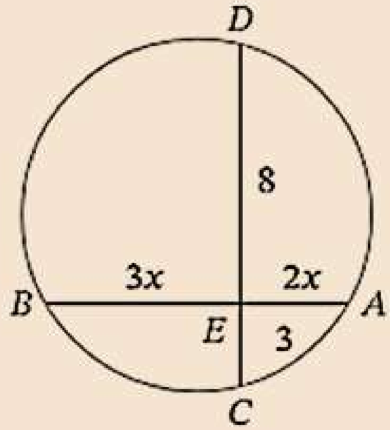
B 20

C 5

D 12.8

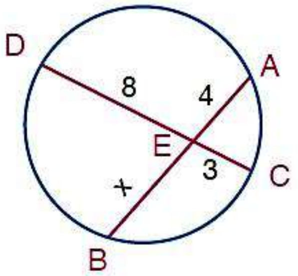
E 5.6

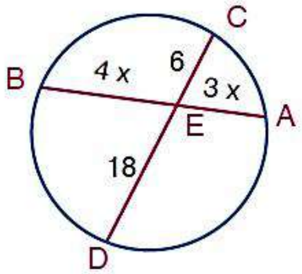
Q8: Find the value of x.

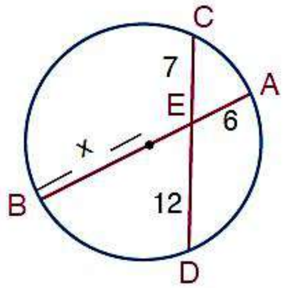


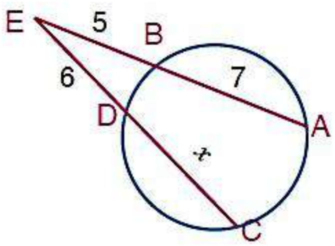
2 - 4 Applications of similarity in the circle

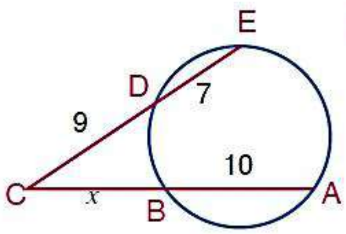
① Use the calculator or mental math to find the numerical value of x in each of the following figures. (lengths are measured in centimetres)

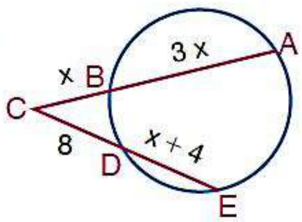
A


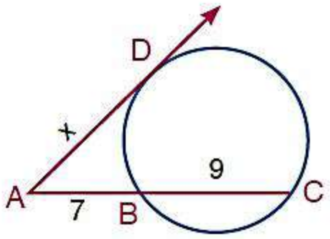
B


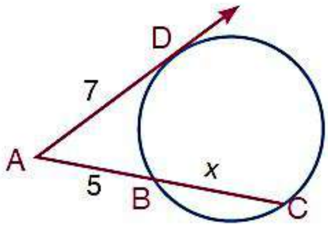
C


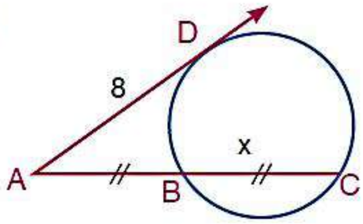
D


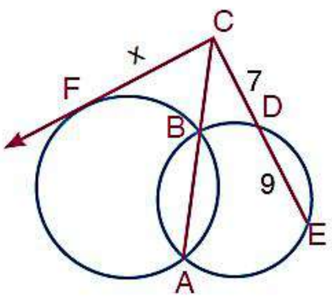
E


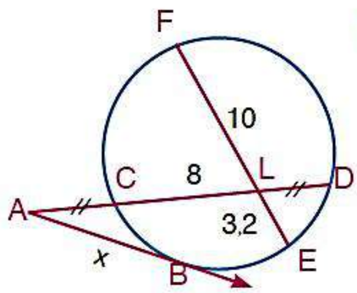
F


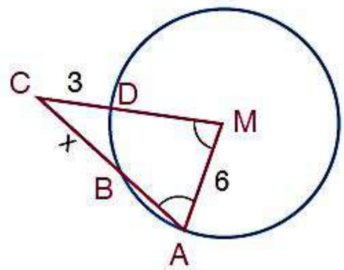
G


H


I


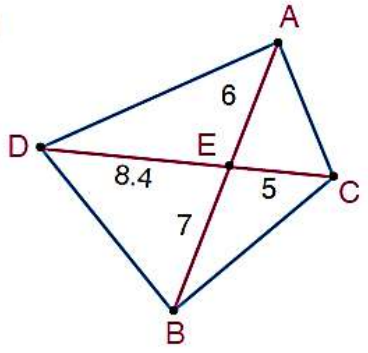
J


K


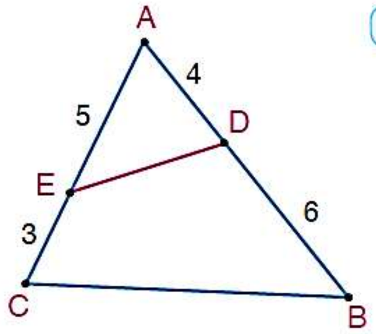
L


(2) In which of the following figures, the points A, B, C and D Lie on a circle? Explain your answer. (the lengths are measured in centimetres)

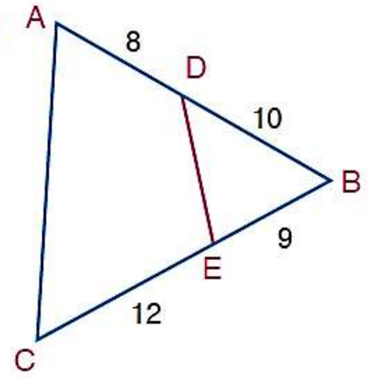
(A)



(B)



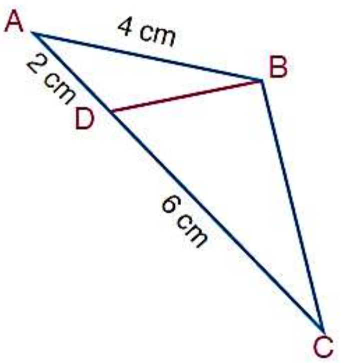
(C)



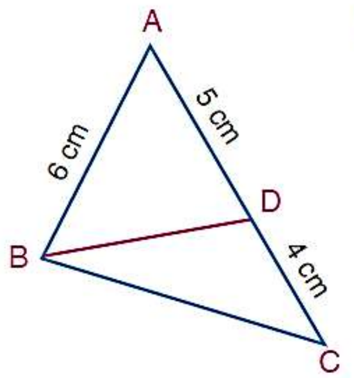
....

(3) In which of the following figures, $\overline{AB}$ is a tangent to the circle passing through the points B, C and D.

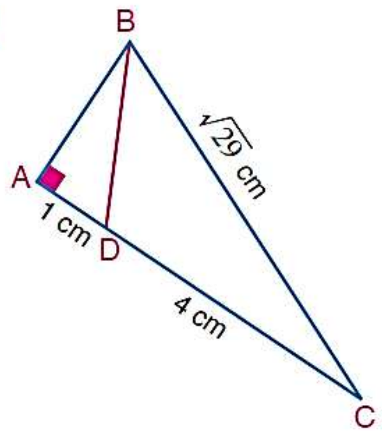
(A)



(B)



(C)

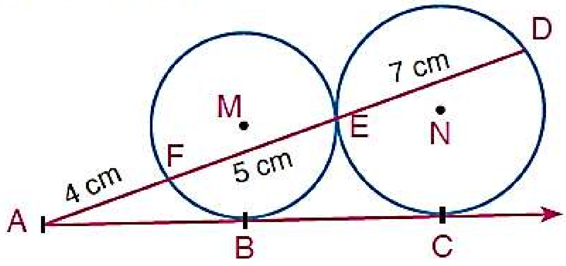


....

(4) Two circles are intersected at A and B. $C \in \overleftrightarrow{AB}$ and $C \notin \overline{AB}$, From C, The two tangent segments $\overline{CX}$ and $\overline{CY}$ are drawn to the circle at X and Y respectively. Prove that $CX = CY$.

....

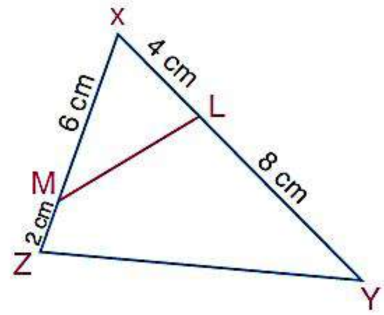
5 **In the figure opposite:** M and N are two tangential circles at E. $\overrightarrow{AC}$ touches the circle M at B, and touches the circle N at C, $\overrightarrow{AE}$ intersects the two circles at F and D respectively,
where AF = 4cm, FE = 5cm, ED = 7cm.
Prove that B is the midpoint of $\overline{AC}$



6 **In the figure opposite:** L ∈ $\overline{XY}$ where XL = 4cm, YL = 8cm , M ∈ $\overline{XZ}$ where XM = 6cm , ZM = 2cm
Prove that:

A $\triangle XLM \sim \triangle XZY$

B LYZM is a cyclic quadrilateral.



7 $\overline{AB} \cap \overline{CE} = \{E\}$, $AE = \frac{5}{12} BE$, $DE = \frac{3}{5} EC$. If BE = 6cm and CE = 5cm.
prove that the points A, B, C and D lie on one circle.

8 ABC is a triangle. D ∈ $\overline{BC}$ where DB = 5cm and DC = 4cm. If AC = 6cm. Prove that:

A $\overline{AC}$ is a tangent segment to the circle passing through the points A, B and D.

B $\triangle ACD \sim \triangle BCA$

C Area of ($\triangle ABD$): area of ($\triangle ABC$) = 5 : 9

9) Two concentric circles at M, their radii are 12cm, 7cm, $\overline{AD}$ is a chord in the larger circle to intersect the smaller circle at B and C respectively. Prove that $AB \times BD = 95$

....

....

....

10) ABCD is a rectangle in which $AB = 6$cm and $BC = 8$cm. $\overrightarrow{BE} \perp \overline{AC}$ and intersects $\overline{AC}$ at E and $\overline{AD}$ at F.

A) Prove that $(AB)^2 = AF \times AD$.

B) Find the length of $\overline{AF}$.

....

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Unit 3

Lesson (1)

Parallel lines and proportional parts

Lesson objectives

Related Links

- Classify Properties of the straight line which is parallel to any side of a triangle.
- Use proportion in calculation of lengths and in prove relations to line segments resulting from the transversals of parallel lines.
- Modeling and solving life problems including parallel lines and their transversals

If a straight Line is drawn parallel to one side of a triangle cutting the other two sides, then it divides them into four segments of proportional lengths.

If ABC is triangle and $\overline{DE} \mathbin{//} \overline{BC}$

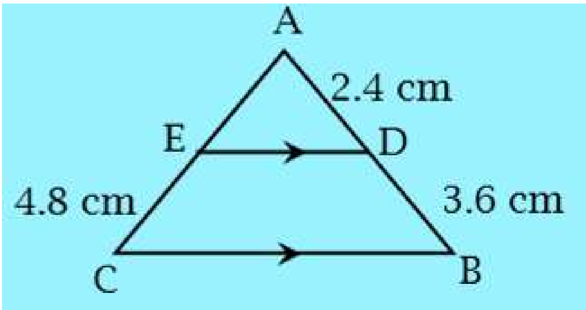
$\therefore \frac{AD}{DB} = \frac{AE}{EC}$ $\qquad \frac{AD}{AB} = \frac{AE}{AC}$ $\qquad \frac{AB}{BD} = \frac{AC}{CE}$

Example ❶

In opposite figure:

ABC is a triangle, $\overline{DE} \mathbin{//} \overline{BC}$, AD = 2.4 cm, DB = 3.6 cm and EC = 4.8 cm.

Find: the length of $\overline{AE}$



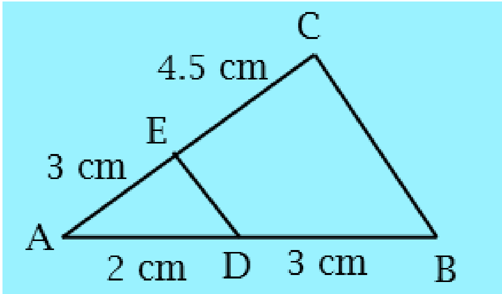
Solution

In Δ ABC:

$\because \overline{DE} \mathbin{//} \overline{BC}$ $\qquad \therefore \frac{AD}{DB} = \frac{AE}{EC}$

$\therefore \frac{2.4}{3.6} = \frac{AE}{4.8}$ $\qquad \therefore AE = \frac{4.8 \times 2.4}{3.6} = 3.2$ cm

Example ❷



In opposite figure:

ABC is a triangle, where AE = 3 cm,

EC = 4.5 cm, AD = 2 cm and DB = 3 cm

Prove that: $\overline{DE} \parallel \overline{BC}$

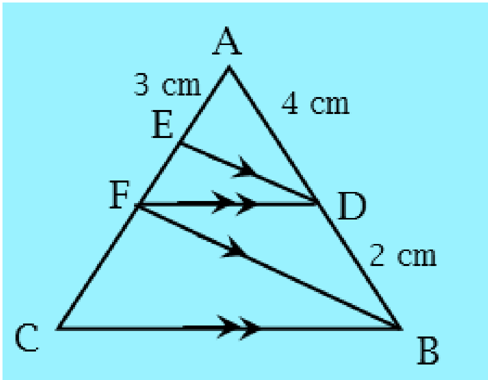
Solution

In Δ ABC:

$\because \frac{AD}{DB} = \frac{2}{3}$ $\because \frac{AE}{EC} = \frac{3}{4.5} = \frac{2}{3}$

$\therefore \frac{AD}{DB} = \frac{AE}{EC}$ $\therefore \overline{DE} \parallel \overline{BC}$

Example ❸



In opposite figure:

ABC is a triangle in which

$\overline{FD} \parallel \overline{BC}$ and $\overline{ED} \parallel \overline{BF}$, AE = 3 cm,

AD = 4 cm and DB = 2 cm

Find the length of: $\overline{EF}$ and $\overline{FC}$

Solution

$\because \overline{ED} \parallel \overline{BF}$ $\therefore \frac{AD}{DB} = \frac{AE}{EF}$

$\therefore \frac{4}{2} = \frac{3}{EF}$ $\therefore EF = \frac{3 \times 2}{4} = 1.5$ cm

$\because \overline{FD} \parallel \overline{BC}$ $\therefore \frac{AD}{DB} = \frac{AF}{FC}$

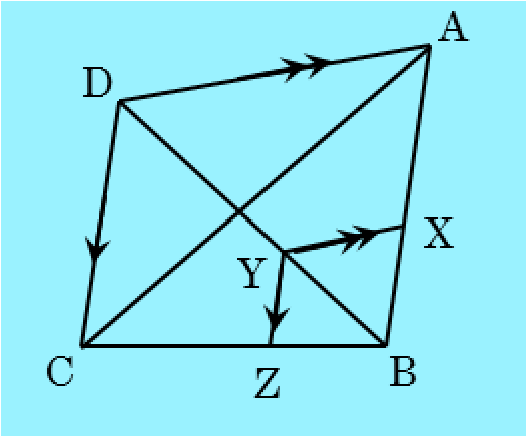
$\therefore \frac{4}{2} = \frac{4.5}{FC}$ $\therefore FC = \frac{4.5 \times 2}{4} = 2.25$ cm

Example ❹

ABCD is a quadrilateral, Y ∈ $\overline{BD}$, $\overline{YX}$ is drawn parallel to $\overline{DA}$ to intersect $\overline{AB}$ at X & $\overline{YZ}$ is drawn parallel to $\overline{DC}$ to intersect $\overline{BC}$ at Z.

Show that: $\overline{XZ}$ // $\overline{AC}$

Solution



In Δ ABD:

$\because \overline{YX} \,//\, \overline{DA}$ $\quad\therefore \frac{BX}{XA} = \frac{BY}{YD}$ ①

In Δ BCD:

$\because \overline{YZ} \,//\, \overline{DC}$ $\quad\therefore \frac{BZ}{ZC} = \frac{BY}{YD}$ ②

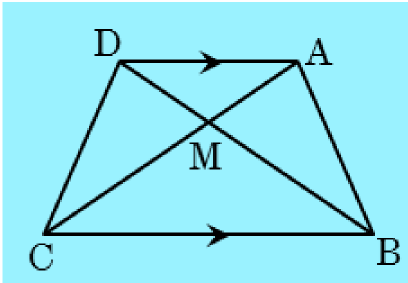
From ① & ②:

$\therefore \frac{BX}{XA} = \frac{BZ}{ZC}$ $\quad\therefore \overline{XZ} \,//\, \overline{AC}$

Example ❺

ABCD is a trapezium in which $\overline{AD}$ // $\overline{BC}$ and $\overline{AC} \cap \overline{BD}$ = { M }
If AM = 2.5 cm , BD = $\frac{22}{3}$ cm , MC = 3 cm and AD = 4.5 cm ,
Then find the length of: $\overline{MD}$, $\overline{MB}$ and $\overline{BC}$

Solution



$\because \overline{AD} \,//\, \overline{BC}$ $\quad\therefore \frac{MD}{BD} = \frac{AM}{AC}$

$\therefore \frac{MD}{\frac{22}{3}} = \frac{2.5}{5.5}$ $\quad\therefore MD = \frac{\frac{22}{3} \times 2.5}{5.5} = \frac{10}{3}$ cm

$\therefore BM = \frac{22}{3} - \frac{10}{3} = 4$ cm

In Δ ADM & Δ CBM:

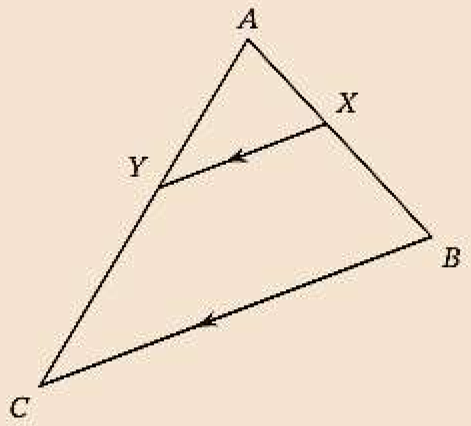
$\because \overline{AD} \,//\, \overline{BC}$ $\quad\therefore \Delta\, ADM \sim \Delta\, CBM$

$\therefore \frac{AD}{CB} = \frac{DM}{BM} = \frac{AM}{CM}$ $\quad\therefore \frac{4.5}{CB} = \frac{2.5}{3}$ $\quad\therefore CB = 5.4$ cm

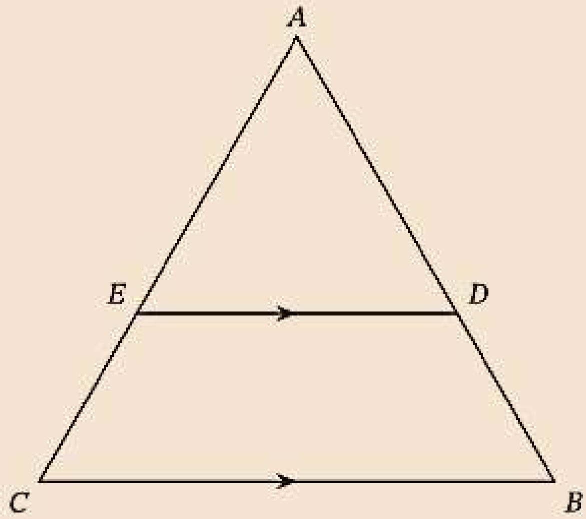
PRACTICE (1)

Q1: In the figure, segments $\overline{XY}$ and $\overline{BC}$ are parallel.

If $AX = 18$, $XB = 24$, and $AY = 27$, what is the length of $\overline{YC}$?



Q2: Determine $\frac{AB}{BD}$, if $\frac{AD}{DB} = \frac{38}{23}$.



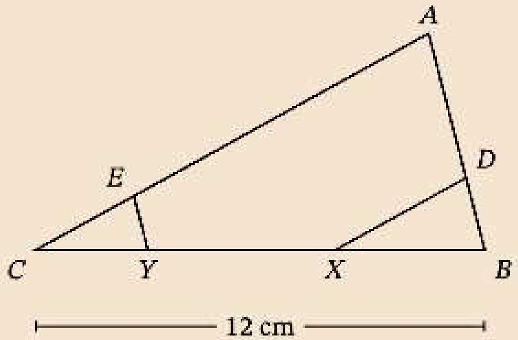
A) $\frac{61}{23}$

B) $\frac{61}{38}$

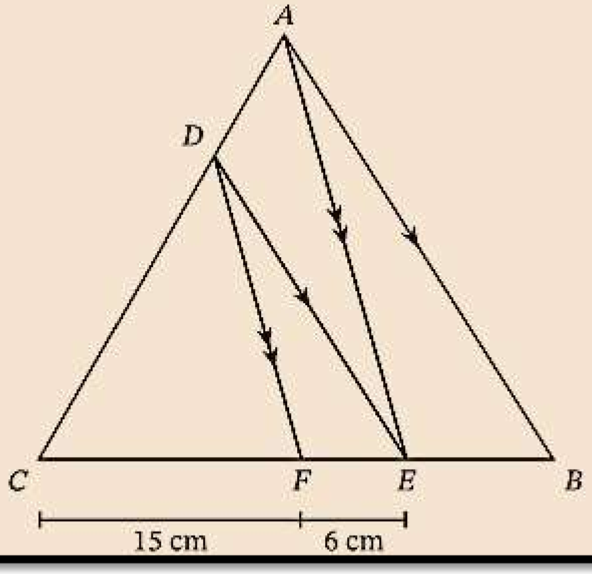
C) $\frac{38}{61}$

D) $\frac{23}{61}$

Q3: In the figure, $\overline{DX}$ and $\overline{EY}$ are parallel to $\overline{AC}$ and $\overline{AB}$ respectively. If $BC = 12$ cm, $\frac{AD}{DB} = 2$, and $EC = \frac{1}{3}AE$, determine the length of $\overline{XY}$.

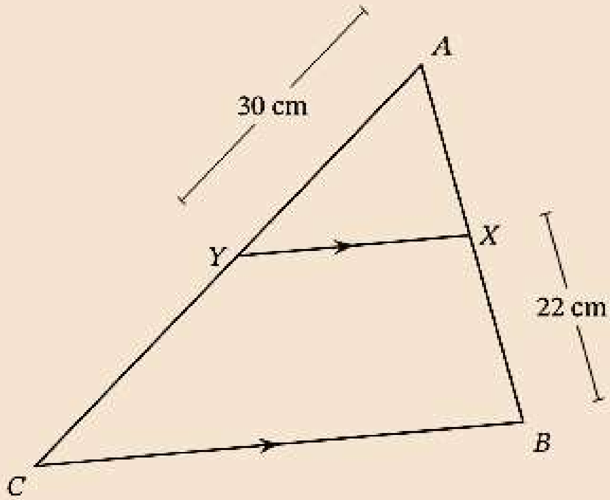


Q4: Find the length of $\overline{CB}$.



Q5: If $BX = 22$ cm, $AY = 30$ cm, and

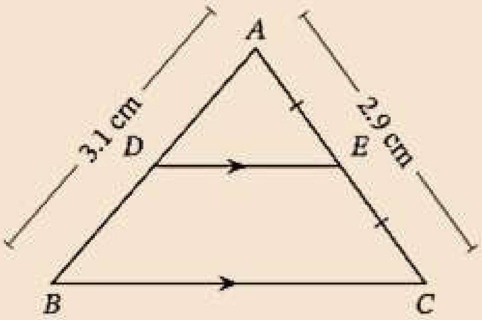
$\dfrac{AX + AY}{AB + AC} = \dfrac{10}{21}$, find the length of $\overline{CY}$.



Q6: If the perimeter of $\triangle ABC = 9.7$ cm,

E is the midpoint of $\overline{AC}$, and $\overline{DE} /\!/ \overline{BC}$,

find the length of $\overline{DE}$.

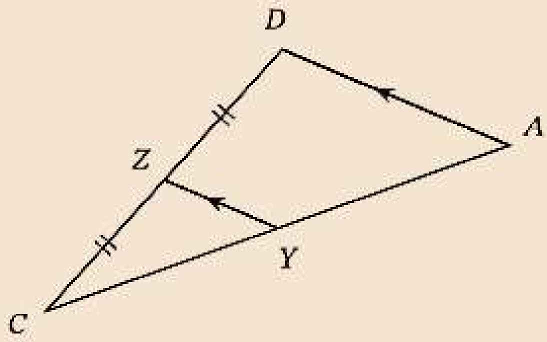


Q7: Given that Z is the midpoint of $\overline{DC}$,

the perimeter of $\triangle ADC$ is 33 cm,

$AD = 7$ cm, and $ZC = 5$ cm,

find the length of $\overline{AY}$.



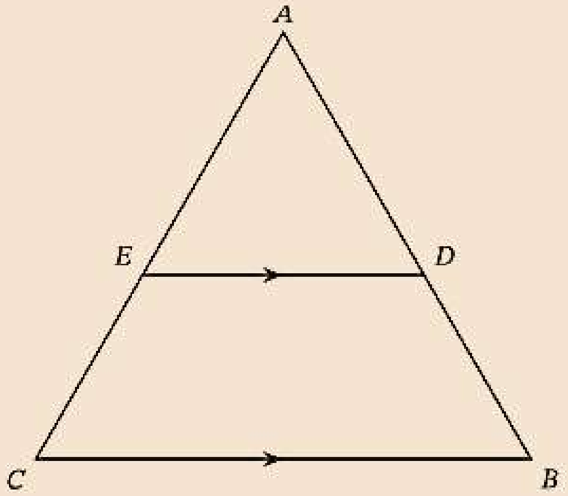
Q8: Determine $\frac{CE}{EA}$, if $\frac{AD}{DB} = \frac{29}{22}$.

A) $\frac{22}{29}$

B) $\frac{51}{29}$

C) $\frac{29}{51}$

D) $\frac{29}{22}$



Q9: In the figure, $AD = 10$, $AE = 5$, $BD = 4x + 2$, and $EC = x + 3$. What are the lengths of $\overline{AB}$ and $\overline{EC}$?

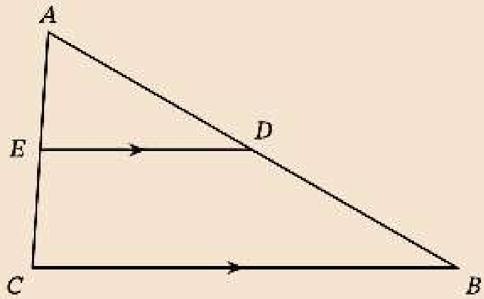
A) $AB = 6, EC = 9$

B) $AB = 16, EC = 4$

C) $AB = 20, EC = 10$

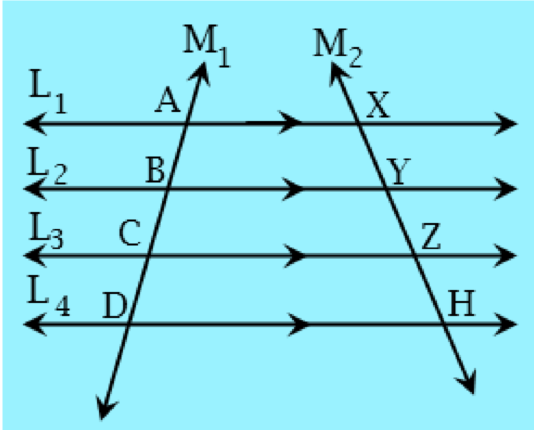
D) $AB = 10, EC = 5$

E) $AB = 20, EC = 5$



Talis Theorem

If two transversals cuts several parallel Lines, so the lengths of the corresponding segments on the two transversals are proportional.



If L_1 // L_2 // L_3 // L_4

And M_1, M_2 are two transversals

$\therefore \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZH}$

OR

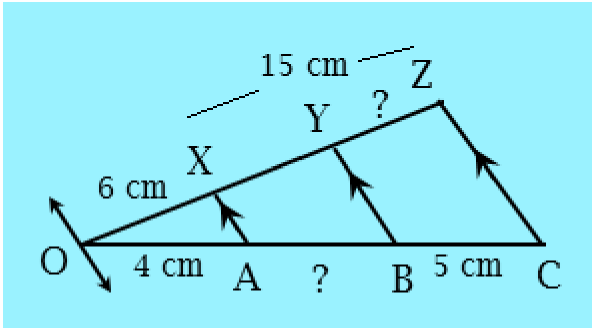
$\therefore \frac{AB}{XY} = \frac{AC}{XZ} = \frac{BD}{YH} = \ldots\ldots\ldots\ldots\ldots$

Example ❶

In the opposite figure:

$\overline{AX}$ // $\overline{BY}$ // $\overline{CZ}$, OX = 6 cm, OA = 4 cm , BC = 5 cm and XZ = 15 cm

Find the length of $\overline{YZ}$, $\overline{AB}$



Solution

$\because \overline{AX}$ // $\overline{BY}$ // $\overline{CZ}$

$\therefore \frac{OX}{YZ} = \frac{OA}{BC}$

$\therefore \frac{6}{YZ} = \frac{4}{5}$

$\therefore YZ = \frac{6 \times 5}{4} = 7.5$ cm

$\therefore XY = 15 - 7.5 = 7.5$ cm

$\because \overline{AX}$ // $\overline{BY}$ // $\overline{CZ}$

$\therefore \frac{OA}{AB} = \frac{OX}{XY}$

$\therefore \frac{4}{AB} = \frac{6}{7.5}$

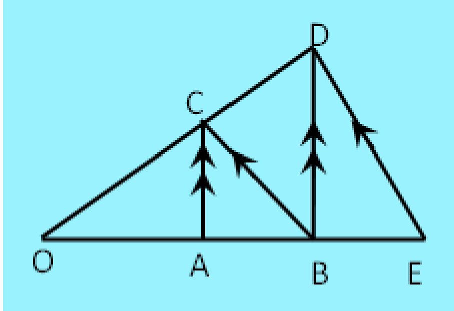
$\therefore AB = \frac{4 \times 7.5}{6} = 5$ cm

Example ❷

In the opposite figure:

$\overline{ED} \parallel \overline{BC}$, $\overline{AC} \parallel \overline{BD}$

Prove that: $(OB)^2 = OA \cdot OE$



Solution

In Δ ABC:

$\because \overline{ED} \parallel \overline{BC}$ $\quad \therefore \frac{OB}{OE} = \frac{OC}{OD}$ ①

$\because \overline{AC} \parallel \overline{BD}$ $\quad \therefore \frac{OA}{OB} = \frac{OC}{OD}$ ②

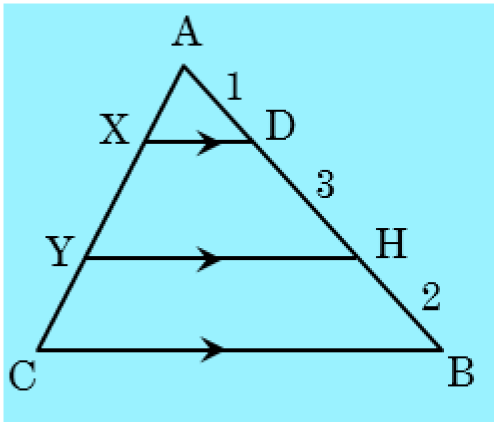
From ① & ②:

$\therefore \frac{OB}{OE} = \frac{OA}{OB}$ $\quad \therefore (OB)^2 = OA \cdot OE$

Example ❸

ABC is a triangle, D & H ∈ $\overline{AB}$ Let $\overrightarrow{DX}$ and $\overrightarrow{HY}$ be drawn parallel to $\overline{BC}$ and intersect $\overline{AC}$ at X & Y respectively. If AD = $\frac{1}{2}$ BH, DH = 3AD & AC = 24 cm , then find the length of each of: $\overline{AX}$, $\overline{XY}$ and $\overline{YC}$

Solution



In Δ ABC:

$\because \overline{DX} \parallel \overline{HY} \parallel \overline{BC}$

$\therefore \frac{AD}{AB} = \frac{AX}{AC}$ $\quad \therefore \frac{1}{6} = \frac{AX}{24}$

$\therefore AX = \frac{24 \times 1}{6} = 4$ cm

$\because \frac{AD}{DH} = \frac{AX}{XY}$ $\quad \therefore \frac{1}{3} = \frac{4}{XY}$

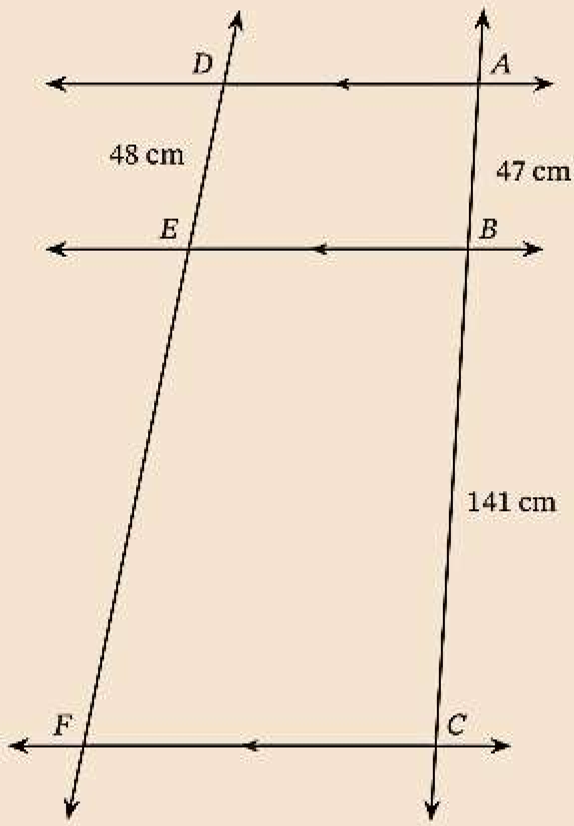
$\therefore XY = \frac{4 \times 3}{1} = 12$ cm

$\therefore YC = 24 - (12 + 4) = 8$ cm

PRACTICE (2)

Q1: Using the information in the figure,

determine the length of $\overline{EF}$.



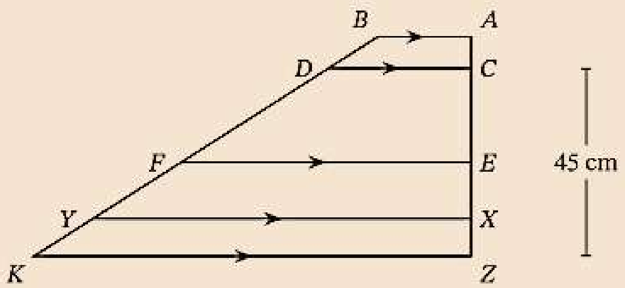
Q2: Given that $AC = 7.5$ cm, $BD = 14$ cm, $FY = 25.2$ cm, and $FK = 42$ cm, determine the lengths of $\overline{CX}$ and $\overline{DF}$.

A) 36 cm, 22.5 cm

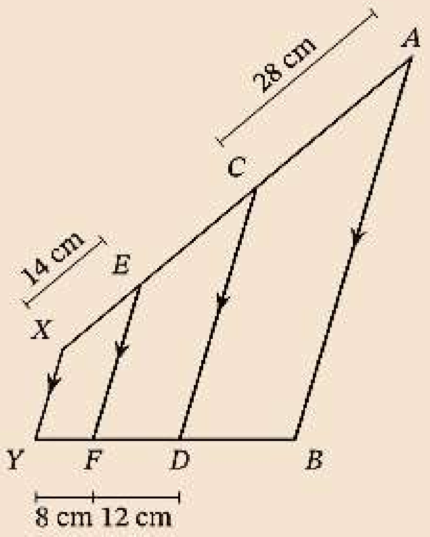
B) 36 cm, 42 cm

C) 67.2 cm, 22.5 cm

D) 37.5 cm, 42 cm



Q3: Find the lengths of $\overline{EC}$ and $\overline{DB}$.



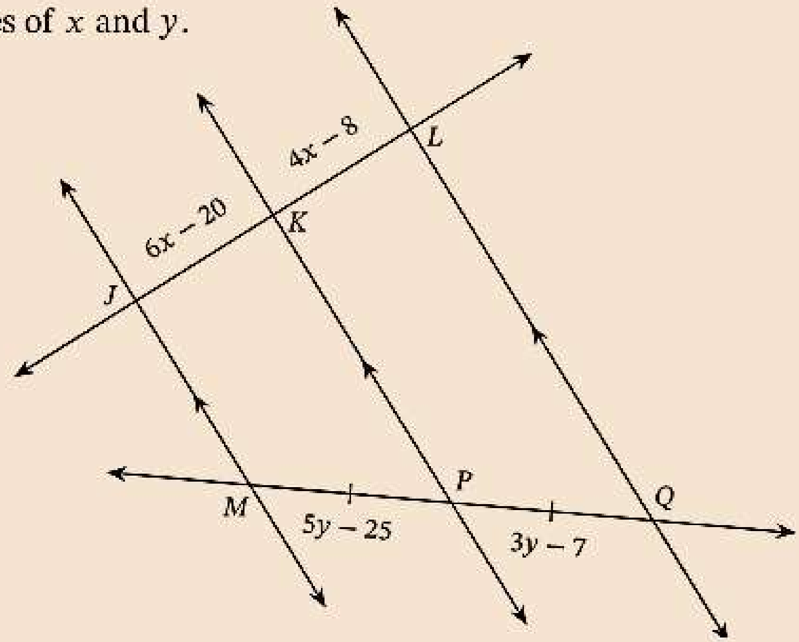
A $EC = 21$ cm, $DB = 16$ cm

B $EC = 28$ cm, $DB = 21$ cm

C $EC = 14$ cm, $DB = 24$ cm

D $EC = 16$ cm, $DB = 24$ cm

Q4: In the given figure, find the values of x and y.



A $x = 3, y = 6$

B $x = 6, y = 3.6$

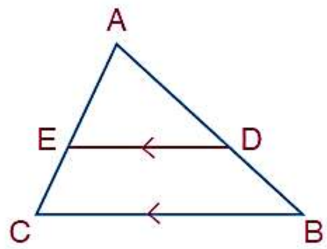
C $x = 12, y = 18$

D $x = 6, y = 9$

E $x = 2, y = 9$

3 - 1 Parallel lines and proportional parts

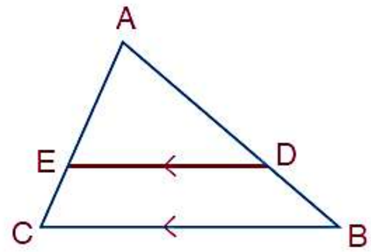
(1) In the figure opposite: $\overline{ED} \mathbin{/\!/} \overline{BC}$. Complete:



(A) if $\frac{AD}{DB} = \frac{5}{3}$ then : $\frac{AB}{BD} = \frac{\dots\dots}{\dots\dots}$ and $\frac{CE}{ED} = \frac{\dots\dots}{\dots\dots}$

(B) if $\frac{AE}{AC} = \frac{4}{7}$, then : $\frac{CE}{EA} = \frac{\dots\dots}{\dots\dots}$ and $\frac{BD}{AB} = \frac{\dots\dots}{\dots\dots}$

(2) In the figure opposite: $\overline{DE} \mathbin{/\!/} \overline{BC}$. Determine the correct statements in each of the following:



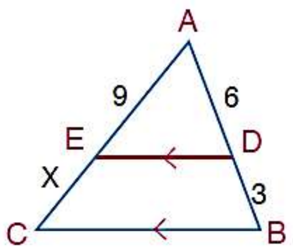
(A) $\frac{AB}{DB} = \frac{AE}{EC}$ (B) $\frac{AD}{AE} = \frac{BD}{EC}$

(C) $\frac{AB}{BD} = \frac{AC}{AE}$ (D) $\frac{AB}{BD} = \frac{AC}{CE}$

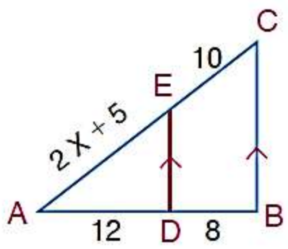
(E) $\frac{AC}{AD} = \frac{AB}{AE}$ (F) $\frac{CE}{BD} = \frac{AC}{AB}$

(3) In each of the following figures: $\overline{DE} \mathbin{/\!/} \overline{BC}$. Find the numerical value of x (length in centimetres).

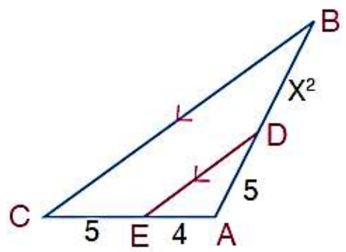
(A)



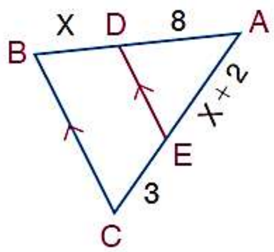
(B)



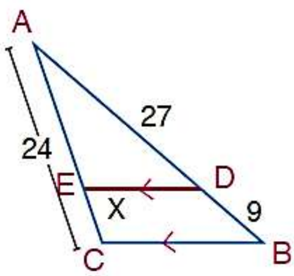
(C)



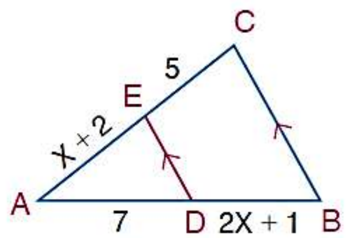
(D)



(E)



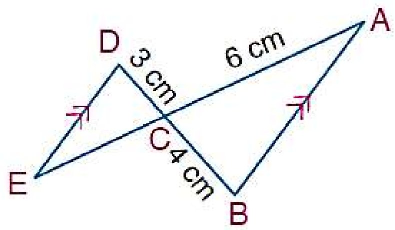
(F)



4 **In the figure opposite:** $\overline{AB} \parallel \overline{DE}$ and $\overline{AE} \cap \overline{BD} = \{C\}$

AC = 6cm, BC = 4cm and CD = 3cm.

Find the length $\overline{AE}$



5 $\overline{XY} \cap \overline{ZL} = \{M\}$, where $\overline{XZ} \parallel \overline{LY}$. If XM = 9cm, Y M = 15cm and ZL = 36 cm, find the length of $\overline{ZM}$.

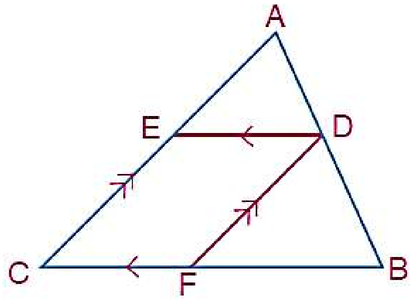
6 For each of the following, use the figure opposite and the given data to find the value of x

A AD = 4 , BD = 8 , CE = 6 and AE = x.

B AE = x , EC = 5 , AD = $x - 2$ and AD = 3.

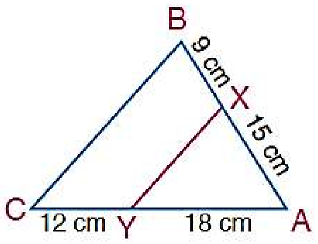
C AB = 21 , BF = 8 , FC = 6 and AD = x.

D AD = x , BF = $x + 5$ and 2DB = 3FC = 12.

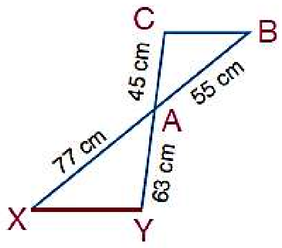


7 In each of the following figures, Is $\overline{XY} \parallel \overline{BC}$?

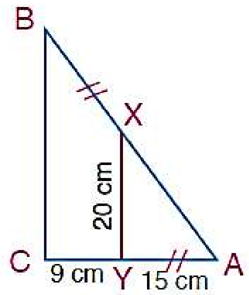
A



B



C



8 XYZ is a triangle in which XY= 14cm, XZ = 21cm and L ∈ $\overline{XY}$ where XL = 5,6cm and M ∈ $\overline{XZ}$ where XM = 8.4cm. Prove that $\overline{LM} \parallel \overline{YZ}$

9 In the triangle ABC, D ∈ $\overline{AB}$, E ∈, $\overline{AC}$ and 5AE = 4 EC.

If AD = 10 cm and DB = 8cm. Is $\overline{DE} \parallel \overline{BC}$? Explain your answer.

10 ABCD is a cyclic quadrilateral, its diagonals are intersected at E. If AE = 6cm, BE = 13cm, EF = 10cm and ED = 7.8 cm. prove that ABCD is a trapezium.

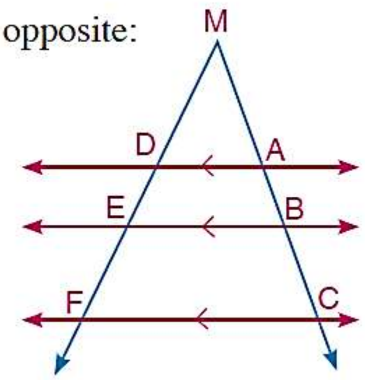
11 Prove that the line segment drawn between two mid points of two sides in a triangle is parallel to the third side and its length is equal to a half of this side.

12 ABC is a triangle, D ∈ $\overline{AB}$ where 3 AD = 2 DB and E ∈ $\overline{AC}$ where 5 C E = 3 A C and $\overline{AX}$ is drawn to intersect $\overline{BC}$ at X. If AF = 8cm and AX = 20cm where F ∈ $\overline{AX}$. Prove that the points D, F and E are collinear.

13 ABC is a triangle, D ∈ $\overline{BC}$, where $\frac{BD}{DC} = \frac{3}{4}$ and E ∈ $\overline{AD}$, where $\frac{AE}{AD} = \frac{3}{7}$. $\overrightarrow{AE}$ is drawn to intersect $\overline{AB}$ at X, $\overrightarrow{DY}$ // $\overline{CX}$ and intersects $\overline{AB}$ at Y. Prove that AX = BY.

14 ABCD is a rectangle, its diagonals are intersected at M. E is the mid point of $\overline{AM}$, F is the midpoint of $\overline{MC}$. $\overrightarrow{DE}$ is drawn to intersect $\overline{AB}$ at X and $\overrightarrow{DF}$ is drawn to intersect $\overline{BC}$ at Y. Prove that: $\overline{XY}$ // $\overline{AC}$.

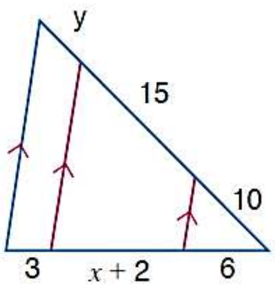
15 Write what each of the following ratios equals using the figure opposite:



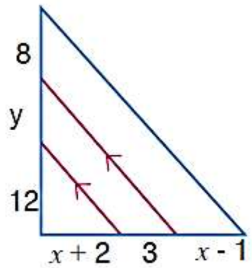
- **A** $\frac{AB}{BC} = \frac{DE}{\cdots\cdots}$
- **B** $\frac{AC}{BC} = \frac{\cdots\cdots}{EF}$
- **C** $\frac{MA}{AB} = \frac{MD}{\cdots\cdots}$
- **D** $\frac{AC}{AB} = \frac{\cdots\cdots}{DE}$
- **E** $\frac{MB}{AB} = \frac{\cdots\cdots}{DE}$
- **F** $\frac{MC}{AC} = \frac{MF}{\cdots\cdots}$
- **G** $\frac{BC}{MB} = \frac{EF}{\cdots\cdots}$
- **H** $\frac{DF}{MF} = \frac{AC}{\cdots\cdots}$

16 In each of the following figures, calculate the numerical values of x and y (lengths are measured in centimetres)

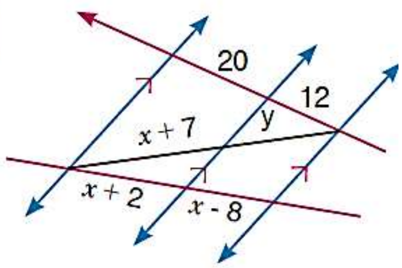
A



B



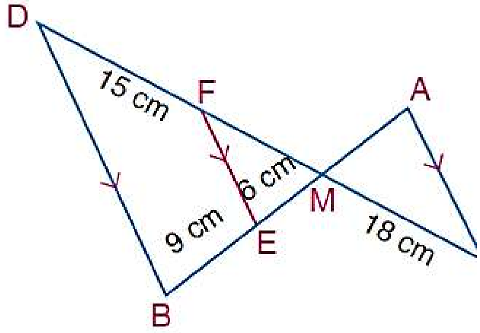
C



17 **In the figure opposite:**

$\overline{AB} \cap \overline{CD} = \{M\}, E \in \overline{MB}$,

$F \in \overline{MD}$ and $\overline{AC} \parallel \overline{FE} \parallel \overline{DB}$.



Find:

A The length of $\overline{MF}$.

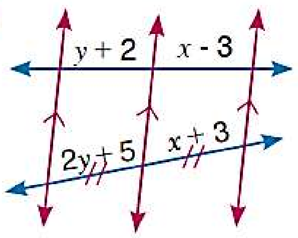
B The length of $\overline{AM}$.

18 $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{E\}, x \in \overline{AB}, y \in \overline{CD}$ and $\overline{XY} \parallel \overline{BD} \parallel \overline{AC}$

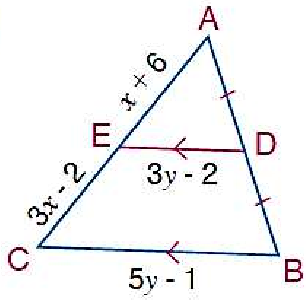
Prove that: $AX \times ED = CY \times EB$.

19 In each of the following figures, calculate the numerical values of x and y:

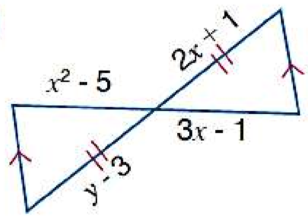
A



B



C



20 ABCD is a quadrilateral in which $\overline{AB} \parallel \overline{CD}$, its diagonals intersect at M and E is the mid point of $\overline{BC}$. $\overrightarrow{EF} \parallel \overline{BA}$ and intersects $\overline{BD}$ at X, $\overline{AC}$ at Y and $\overline{AD}$ at F.

prove that:

A $EY = \frac{1}{2} AB$.

B $\frac{AY}{CM} = \frac{BX}{DM}$

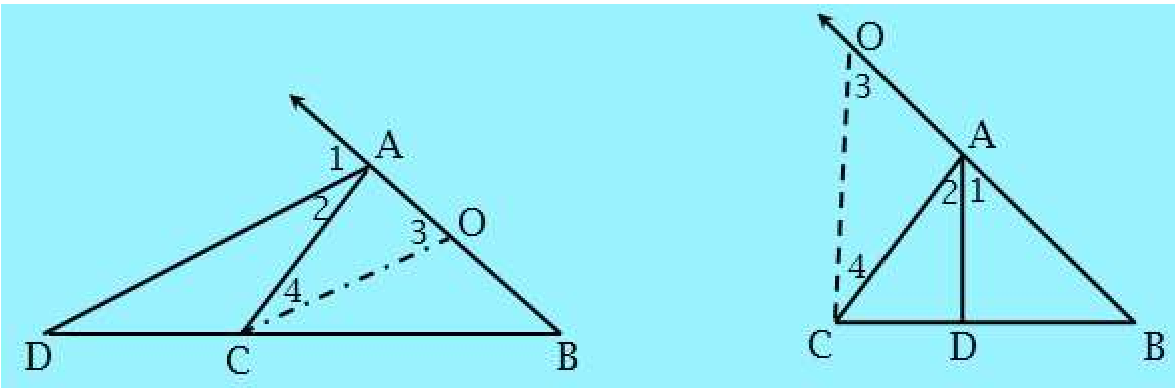
Lesson (2) — Angle Bisectors & proportional parts

Lesson objectives

Related Links

- Classify Properties of bisectors of angles of triangles.
- Use proportion to calculate the lengths of line segments resulting from bisecting an angle in a triangle.
- Modeling and solving life problems including bisectors of angles of triangle.

The bisector of the Interior or (exterior) angle of a triangle at any vertex divided the opposite side of the triangle internally or (externally) into two parts the ratio between their lengths is equal to the ratio between the lengths of other two sides.



Given: $\overline{AD}$ bisects $\angle$ BAC

Construction: Draw $\overline{CO}$ // $\overline{DA}$

R.T.P.: Prove that $\frac{AB}{AC} = \frac{BD}{CD}$

Proof: $\because$ $\overline{CO}$ // $\overline{DA}$

$\therefore$ m ($\angle$ 2) = m ($\angle$ 4) **(Alternative angle)**

& m ($\angle$ 1) = m ($\angle$ 3) **(Corresponding angle)**

$\because$ m ($\angle$ 1) = m ($\angle$ 2) **(Given)**

$\therefore$ m ($\angle$ 3) = m ($\angle$ 4)

$\therefore$ AO = AC

$\therefore \frac{AB}{AO} = \frac{BD}{CD}$

$\because$ $\overline{CO}$ // $\overline{DA}$

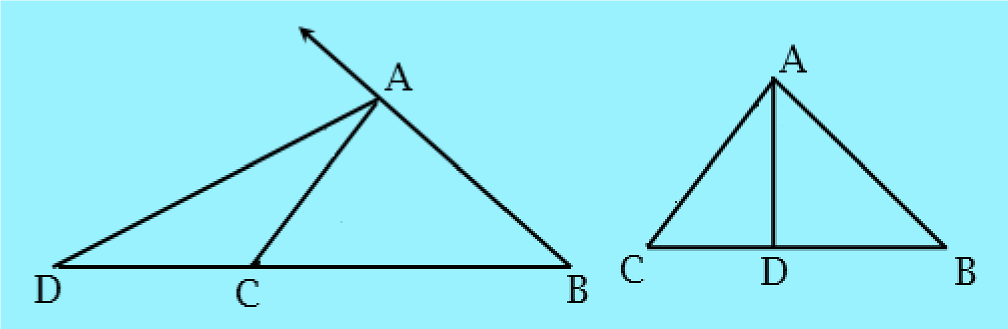
$\therefore \frac{AB}{AO} = \frac{BD}{CD}$

Remarks:

❶ Converse Of Theorem:

If: $\frac{AB}{AC} = \frac{BD}{CD}$

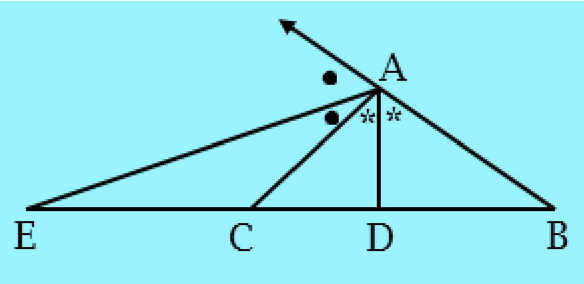
$\therefore \overline{AD}$ bisects $\angle BAC$



❷ The bisector interior and exterior:

① The interior bisector ⊥ The exterior bisector

i.e. $\therefore \overline{AD} \perp \overline{AE}$



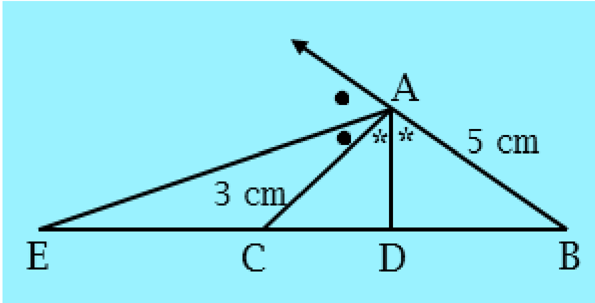
② $\frac{EB}{EC} = \frac{BD}{CD}$

$\because \overrightarrow{AD}$ bisects $\angle BAC$ $\therefore \frac{AB}{AC} = \frac{BD}{CD}$ $\Longrightarrow$ **(i)**

$\because \overrightarrow{AE}$ bisects exterior $\angle BAC$ $\therefore \frac{AB}{AC} = \frac{BE}{CE}$ $\Longrightarrow$ **(ii)**

From (i) & (ii): $\therefore \frac{EB}{EC} = \frac{BD}{CD}$

Example ❶



In the opposite figure:

$\overline{AD}$ bisects ∠ BAC and

$\overline{AE}$ bisects exterior ∠ BAC

AB = 5 cm, AC = 3 cm, BD = 4 cm

Find the length of $\overline{ED}$

Solution

∵ $\overrightarrow{AD}$ bisects ∠ BAC

$\therefore \frac{AB}{AC} = \frac{BD}{CD}$ $\therefore \frac{5}{3} = \frac{4}{CD}$

$\therefore CD = \frac{3 \times 4}{5} = 2.4$ cm

∵ $\overrightarrow{AE}$ bisects exterior ∠ BAC

$\therefore \frac{AB}{AC} = \frac{BE}{CE}$ $\therefore \frac{5}{3} = \frac{BE}{CE}$

$\therefore \frac{5}{3} = \frac{CE + BC}{CE}$

∵ BC = 4 + 2.4 = 6.4 cm $\therefore \frac{5}{3} = \frac{CE + 6.4}{CE}$

∴ 5 CE = 3 (CE + 6.4)

∴ 5 CE = 3 CE + 19.2 ∴ 2 CE = 19.2

$\therefore CE = \frac{19.2}{2} = 9.6$ cm

∴ ED = 9.6 + 2.4 = 12 cm

Well-known problem:

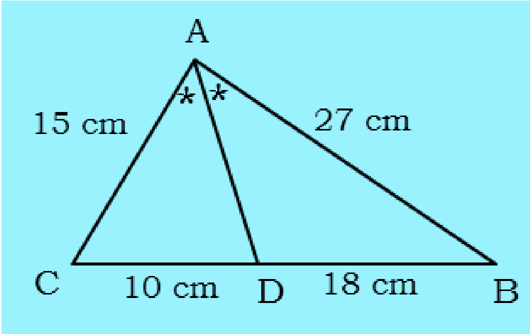
If $\overrightarrow{AD}$ bisects ∠ A in Δ ABC internally and intersects $\overline{BC}$ at D

then: $AD = \sqrt{AB \times AC - BD \times DC}$

Example ❷

ABC is a triangle in which AB = 27 cm, AC = 15 cm. $\overrightarrow{AD}$ bisects ∠ A and intersects $\overline{BC}$ at D. If BD = 18 cm & CD = 10. Calculate the length of $\overline{AD}$.

Solution



∵ $\overrightarrow{AD}$ bisects ∠ A

∴ $AD = \sqrt{AB \times AC - BD \times DC}$

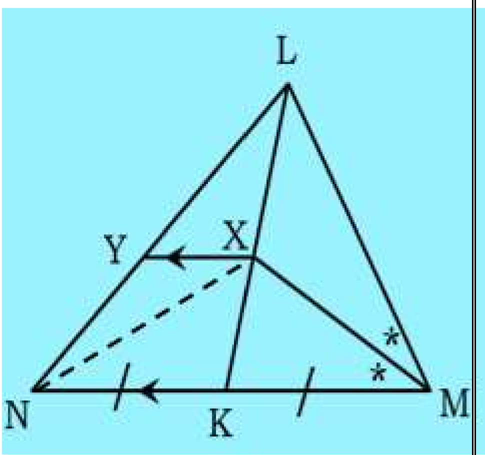
∴ $AD = \sqrt{27 \times 15 - 18 \times 10}$

∴ AD = 15 cm

Example ❸

LMN is a triangle, k is mid-point of $\overline{MN}$, $\overrightarrow{MX}$ bisects ∠ LMN & cuts $\overline{LK}$ at X. Draw $\overrightarrow{XY}$ // $\overline{MN}$ cuts $\overline{LN}$ at Y. If Lk = LM prove that $\overline{KY}$ bisect ∠ LKN

Solution



In Δ LMK:

∵ $\overrightarrow{MX}$ bisects ∠ LMN

∴ $\frac{ML}{MK} = \frac{LX}{KX}$ ①

In Δ LMN:

∵ $\overrightarrow{XY}$ // $\overline{MN}$

∴ $\frac{LX}{XK} = \frac{LY}{YN}$ ②

From ① & ②: ∴ $\frac{ML}{MK} = \frac{LY}{YN}$

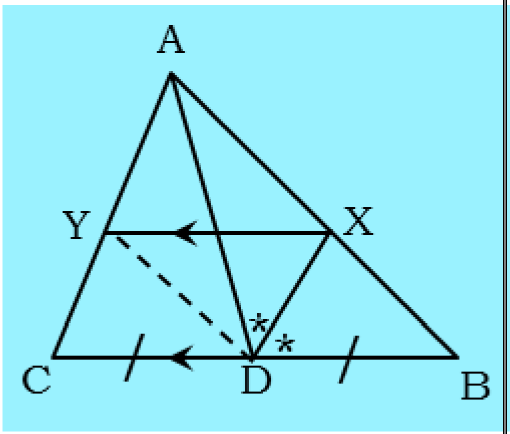
∵ MK = KN & ∵ LK = LM

∴ $\frac{LK}{KN} = \frac{LY}{YN}$ ∴ $\overrightarrow{MX}$ bisects ∠ LMN

Example ❹

$\overline{AD}$ is a median of $\triangle$ ABC, $\angle$ ADB is bisected by a bisector meets $\overline{AB}$ at X and $\overline{XY}$ // $\overline{BC}$ meets $\overline{AC}$ at Y. Prove that $\overline{DY}$ bisects $\angle$ ADC.

Solution



In Δ ABD:

$\because \overrightarrow{DX}$ bisects $\angle$ ADB $\quad \therefore \frac{DA}{DB} = \frac{AX}{BX}$ ①

In Δ ABC:

$\because \overline{XY}$ // $\overline{BC}$ $\quad \therefore \frac{AX}{BX} = \frac{AY}{CY}$ ②

From ① & ②: $\quad \therefore \frac{DA}{DB} = \frac{AY}{CY}$

$\because \overline{AD}$ is a median of Δ ABC $\quad \therefore$ BD = CD

$\therefore \frac{DA}{DC} = \frac{AY}{CY}$ $\quad \therefore \overrightarrow{DY}$ bisects $\angle$ ADC

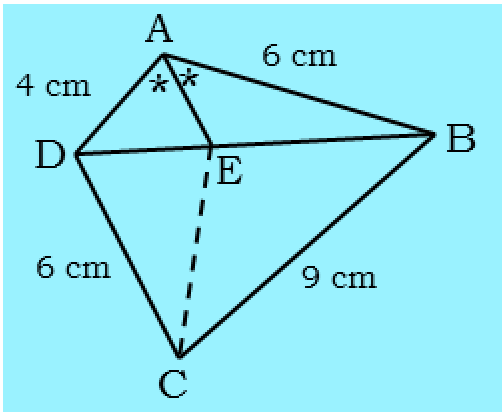
Example ❺

ABCD is a quadrilateral AB = 6 cm, BC = 9 cm, CD = 6 cm, DA = 4 cm, $\overrightarrow{AE}$ bisect $\angle$ A to cut $\overline{BD}$ at E.

(i) Find the ratio BE : ED

(ii) Show that $\overrightarrow{CE}$ bisect $\angle$ BCD

Solution



In Δ ABD:

$\because \overrightarrow{AE}$ bisects $\angle$ BAD $\quad \therefore \frac{AB}{AD} = \frac{BE}{DE}$

$\therefore \frac{BE}{DE} = \frac{6}{4} = \frac{3}{2}$

$\therefore$ BE : ED = 3 : 2 **(i)**

In Δ BCD:

$\because \frac{CB}{CD} = \frac{9}{6} = \frac{3}{2}$ $\quad \therefore \frac{CB}{CD} = \frac{BE}{DE}$

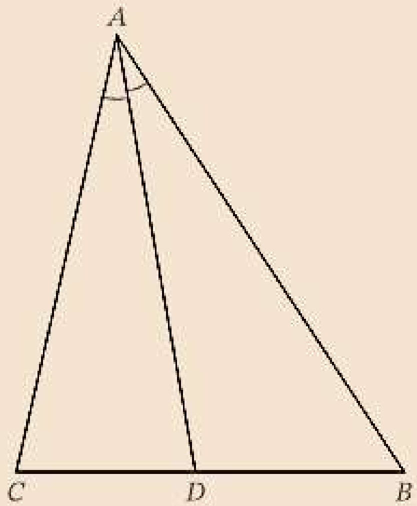
$\therefore \overrightarrow{CE}$ bisects $\angle$ BCD **(ii)**

PRACTICE

Q1: In the given figure, $AB = 35$, $AC = 30$,

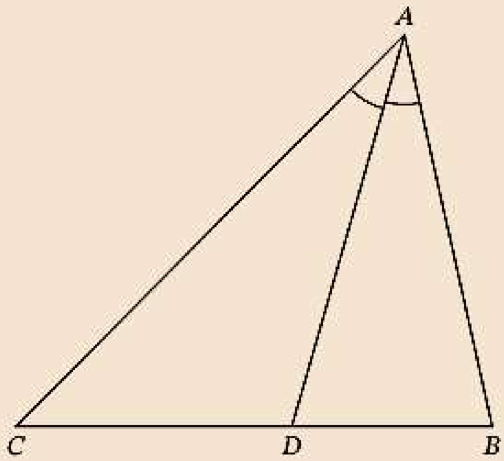
and $CD = 12$. If $BD = x + 10$

, what is the value of x?



Q2: In the figure, $\overline{AD}$ bisects $\angle BAC$, $BD = 8$, $DC = 11$

., and the perimeter of $\triangle ABC$ is 57. Determine the lengths of $\overline{AB}$ and $\overline{AC}$.



A $AB = 22$, $AC = 16$

B $AB = 19$, $AC = 22$

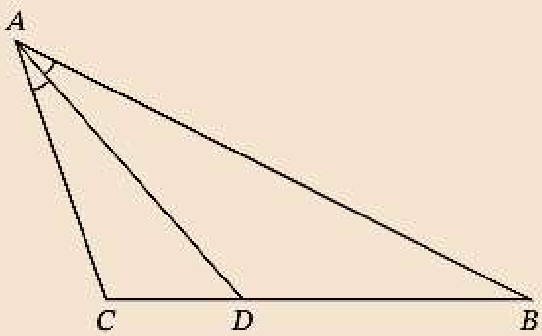
C $AB = 16$, $AC = 19$

D $AB = 16$, $AC = 22$

Q3: Given that angle A is bisected by $\overline{DA}$,

$AB = 38$, $AC = 18$, and $BC = 28$.

Determine the lengths DB and DC.



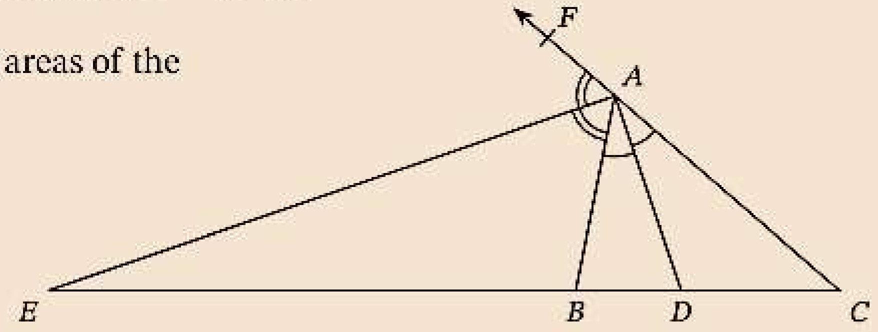
A $DB = 14$, $DC = 14$

B $DB = 9$, $DC = 19$

C $DB = 19$, $DC = 9$

D $DB = 12$, $DC = 16$

Q4: If $AB = 30$ cm, $BC = 40$ cm, and $AC = 45$ cm, find the ratio between the areas of the $\triangle AED$ and the $\triangle AEC$.



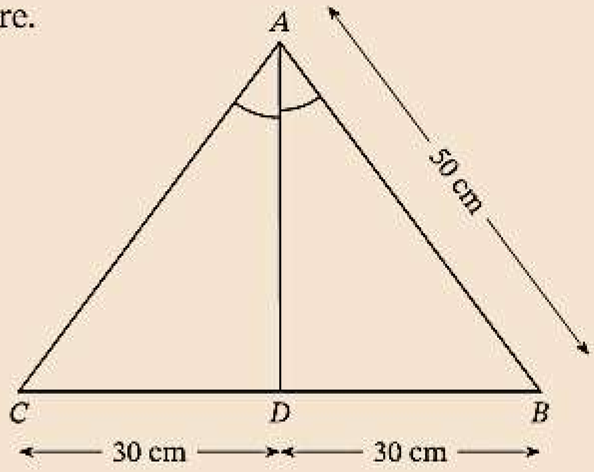
A 4 : 5

B 8 : 9

C 8 : 15

D 3 : 4

Q5: Find the lengths of $\overline{AC}$ and $\overline{AD}$ in the figure.



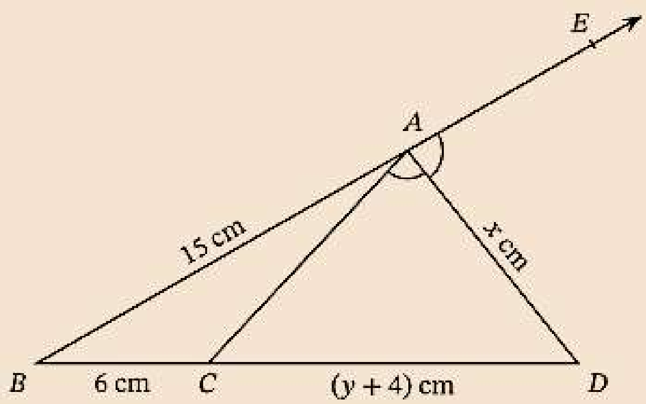
A $AC = 55$ cm, $AD = 50$ cm

B $AC = 55$ cm, $AD = 58$ cm

C $AC = 50$ cm, $AD = 40$ cm

D $AC = 58$ cm, $AD = 55$ cm

Q6: : Given that ABC is a triangle in which $AC = 10$ cm, find the value of each of x and y.



A $x = \sqrt{66}, y = 12$

B $x = \sqrt{66}, y = 8$

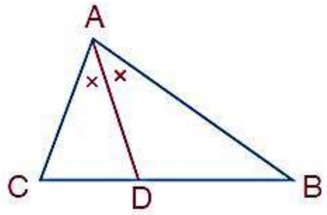
C $x = 12, y = \sqrt{66}$

D $x = 8, y = \sqrt{66}$

Angle Bisectors and Proportional Parts

3 - 2

1 **In the figure opposite:** $\overrightarrow{AD}$ bisects $\angle A$. Complete:



A $\frac{BD}{DC} =$ ....

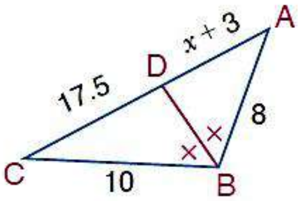
B $\frac{AC}{CB} =$ ....

C $\frac{BD}{BA} =$ ....

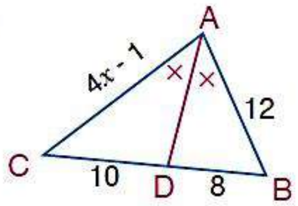
D $AB \times CD =$ ....

2 **In each of the following figures**: find the value of X (lengths are estimated in centimetres)

A

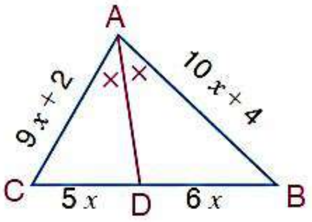


B

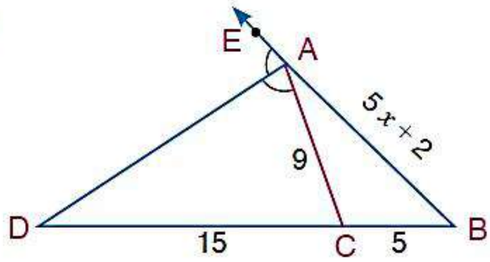


....

C



D



....

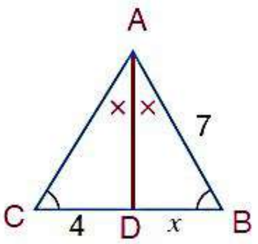
3 ABC is a triangle. its perimeter is 27cm. $\overrightarrow{BD}$ bisects $\angle B$ and intersects $\overline{AC}$ at D.

If AD = 4cm and CD = 5cm, find the length of $\overline{AB}$, $\overline{BC}$ and $\overline{AD}$

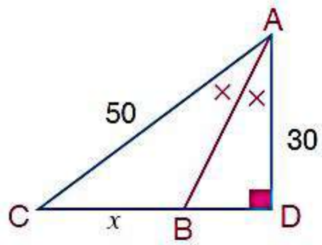
....

4 **In each of the following figures**, find the value of x then find the perimeter of $\triangle ABC$.

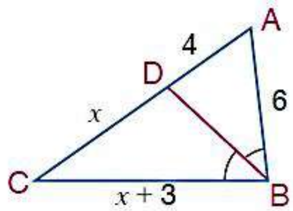
A



B



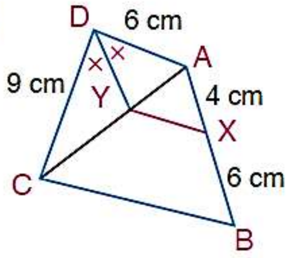
C



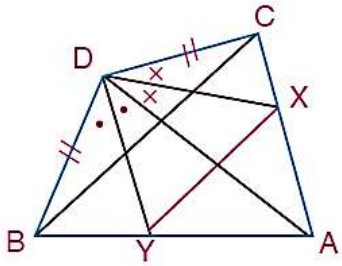
(5) ABC is a triangle in which AB = 8cm, AC = 4cm and BC = 6cm and $\overrightarrow{AD}$ bisects $\angle A$ and intersects $\overline{BC}$ at D and $\overrightarrow{AE}$ bisects the exterior angle at A and intersects $\overleftrightarrow{BC}$ at E. Find the length of $\overline{DE}$, $\overline{AD}$ and $\overline{AE}$.

(6) In each of the following figures, prove that $\overline{XY} \,//\, \overline{BC}$

A



B



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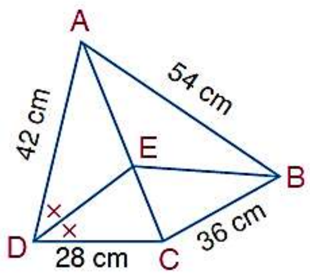
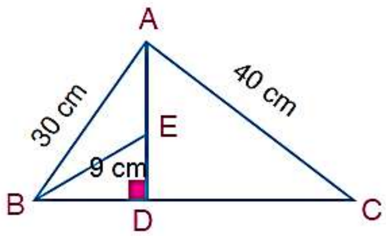
....

(7) In each of the following figures, prove that $\overrightarrow{BE}$ bisects $\angle ABC$.

A



B



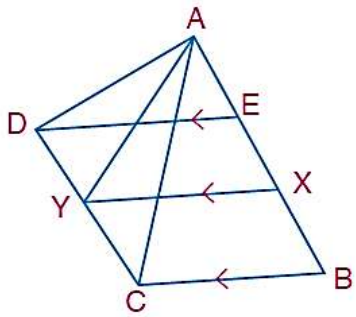
....

....

(8) **In the figure opposite:** $\overline{ED} \,//\, \overline{XY} \,//\, \overline{BC}$ and

$AD \times BX = AC \times EX$.

Prove that $\overrightarrow{AY}$ bisects $\angle CAD$.

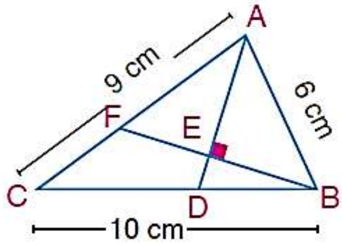


....

....

(9) ABC is a triangle, $D \in \overrightarrow{BC}$, $D \notin \overline{BC}$ where $CD = AB$. $\overrightarrow{CE} \parallel \overline{DA}$ and intersects $\overline{AB}$ at E. $\overrightarrow{EF} \parallel \overrightarrow{BC}$ and intersects $\overline{AC}$ at F. Prove that $\overrightarrow{BF}$ bisects $\angle ABC$.

(10) **In the figure opposite:** ABC is a triangle in which $AB = 6cm$, $AC = 9cm$ and $BC = 10cm$. $D \in \overline{BC}$ where $BD = 4cm$.
$\overrightarrow{BE} \perp \overline{AD}$ and intersects $\overline{AD}$ and $\overline{AB}$ at E and F respectively.



(A) Prove that $\overrightarrow{AD}$ bisects $\angle A$.

(B) Find area of ($\triangle$ ABF) : area of ($\triangle$ CBF)

Lesson (3)

Applications of Proportionality in the Circle

Lesson objectives

Related Links

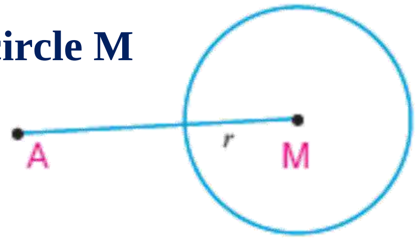
- Find the power of a point w.r.to the circle.
- Determine the position of a point w.r. to the circle.
- Find the measures of the resulting angles from the intersection of chords and tangents in the circle.

First: Power of a point with respect to a circle

Power of the point A w.r. to the circle M whose radius r is the real number $P_M(A)$ where: $P_M(A) = (AM)^2 - r^2$

Important Notes:

❶ You can expect the position of point A w.r. to the circle M

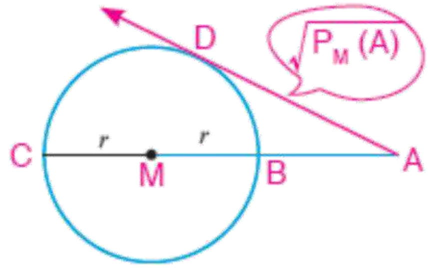


If: $P_M(A) > 0$ then A lies outside the circle.

$P_M(A) = 0$ then A lies on the circle.

$P_M(A) < 0$ then A lies inside the circle.

❷ If the point A lies outside the circle M



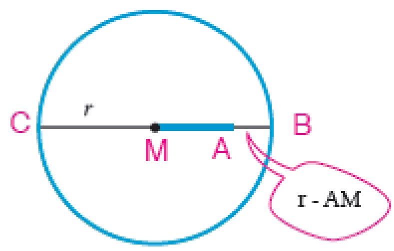
then : $P_M(A) = (AM)^2 - r^2$

$= (AM - r)(AM + r)$

$= AB \times AC = (AD)^2$

$\therefore$ Length of the tangent drawn from A to circle M $= \sqrt{P_M(A)}$

❸ If the point A lies inside the circle M



then: $P_M(A) = (AM)^2 - r^2$

$= (AM - r)(AM + r)$

$= -(r - AM)(AM + r)$

$= -AB \times AC$

Example ❶

Determine the position of each of the following points A, B and C w.r. to the circle M in which its radius equals 5cm, if:

$P_M(A) = 11$, $P_M(B) = 0$, $P_M(C) = -16$, then calculate the distance between each point to the center of the circle.

Solution

$\because P_M(A) = 11 > 0$ $\therefore$ A lies outside the circle

$\because P_M(A) = (AM)^2 - r^2$ $\therefore 11 = (AM)^2 - 25$ $\therefore AM = 6cm$

$\because P_M(B) = 0$ $\therefore$ B lies on the circle $\therefore BM = 5cm$

$\because P_M(C) = -16$ $\therefore$ C lies inside the circle

$\because P_M(C) = (CM)^2 - r^2$ $\therefore -16 = (CM)^2 - 25$ $\therefore CM = 3cm$

Example ❷

The radius of circle M equals 31cm. The point A lies at 23cm distance from its radius center. Draw the chord $\overline{BC}$ where $A \in \overline{BC}$, AB = 3 AC. Calculate:

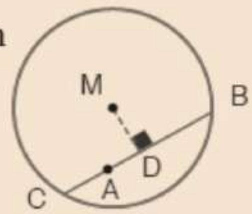
(A) Length of the chord $\overline{BC}$ and the center of the circle.

(B) The distance between the chord $\overline{BC}$

Solution

In the circle M:

(A) $\because r = 31cm, AM = 23cm, A \in \overline{BC}$ $\therefore$ A lies inside the circle, then



$P_M(A) = (AM)^2 - r^2 = -AB \times AC$

$(23)^2 - (31)^2 = -3AC \times AC$ $\therefore AC = 12cm$

$\therefore$ Length of the chord $\overline{BC} = 4AC = 4 \times 12 = 48cm$

(B) let the distance between the chord and the centre of the circle be MD where $\overline{MD} \perp \overline{BC}$

$\because \overline{MD} \perp \overline{BC}$ $\therefore$ D is midpoint of $\overline{BC}$, then BD = 24cm

$\therefore (MD)^2 = (31)^2 - (24)^2 = 385$ $\therefore MD = \sqrt{385} \simeq 19.6cm$

PRACTICE (1)

Q1: A point is at a distance 40 from the centre of a circle. If its power with respect to the circle is 81, what is the radius of the circle, rounded to the nearest integer?

A 21

B 41

C 19

D 39

Q2: A circle with centre M and a mark A satisfy $MA = 28$ cm and $P_M(A) = 4$. Using $\pi = \frac{22}{7}$, find the area and the circumference of the circle to the nearest integer.

A Area $= 88$ cm^2, circumference $= 176$ cm

B Area $= 2\,451$ cm^2, circumference $= 176$ cm

C Area $= 2\,451$ cm^2, circumference $= 88$ cm

D Area $= 4\,903$ cm^2, circumference $= 88$ cm

Q3: A circle has center M and radius $r = 21$. Find the power of the point A with respect to the circle given that $AM = 25$.

Q4: A circle with centre N has a diameter equal to 38 cm. A point B satisfies $NB = 7$ cm. Find the power of B with respect to the circle, giving your answer to the nearest integer.

A $-1\,395$

B -312

C $1\,395$

D 312

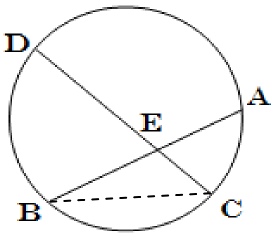
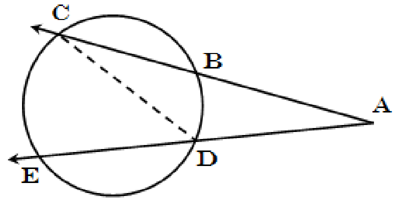
Q5: A circle with centre M has a radius of 8 cm. The power of a mark A with respect to the circle is 36. Decide whether A is outside, inside, or on the circle and then find the distance between A and M.

A On the circle, 28 cm

B Inside the circle, 44 cm

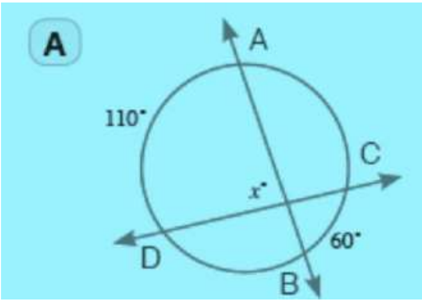
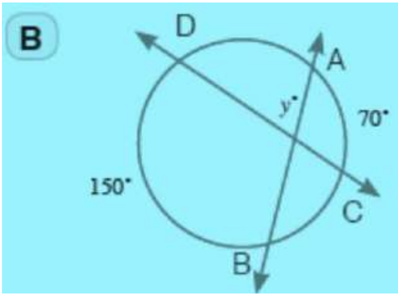
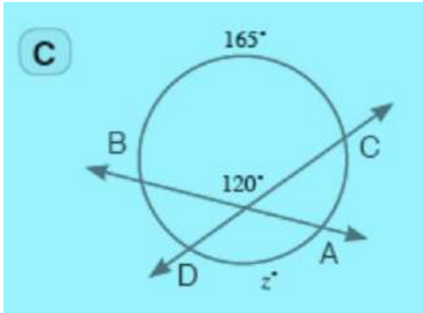
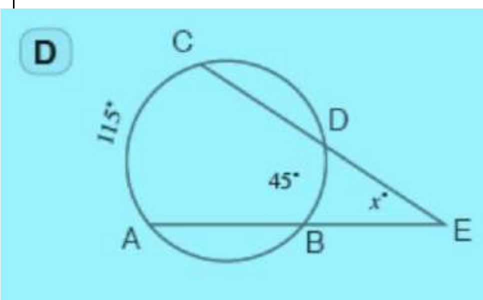
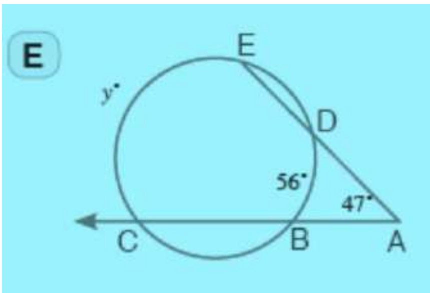
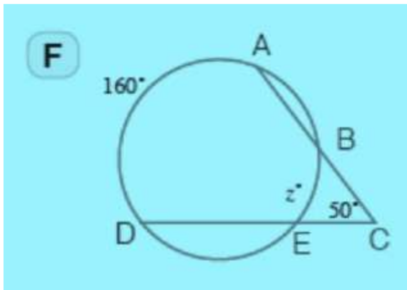
C Outside the circle, 10 cm

Second: secant, tangent, and measures of angles

$m(\angle BEC) = \frac{1}{2}[m(\widehat{BC}) + m(\widehat{AD})]$	$m(\angle A) = \frac{1}{2}[m(\widehat{CE}) - m(\widehat{BD})]$
	

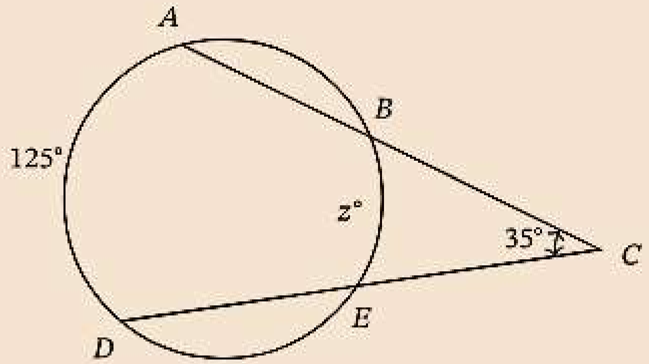
Example ❸

In each of the following figures: Find the value of the symbol:

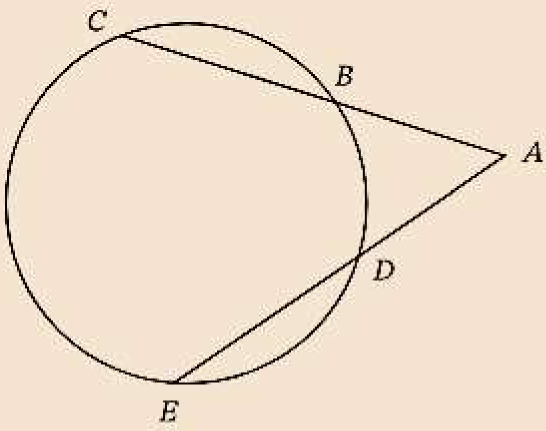
		
$\because x = \frac{1}{2}(110° + 60°)$ $\therefore x = 85°$	$180 - y = \frac{1}{2}(150° + 70°)$ $\therefore y = 180° - 110° = 70°$	$\because z = 2 \times 120° - 165°$ $\therefore z = 75°$
		
$\because x = \frac{1}{2}(115° - 45°)$ $\therefore x = 35°$	$\because y = 2 \times 47° + 56°$ $\therefore y = 150°$	$\because z = 165° - 2 \times 50°$ $\therefore z = 65°$

PRACTICE (2)

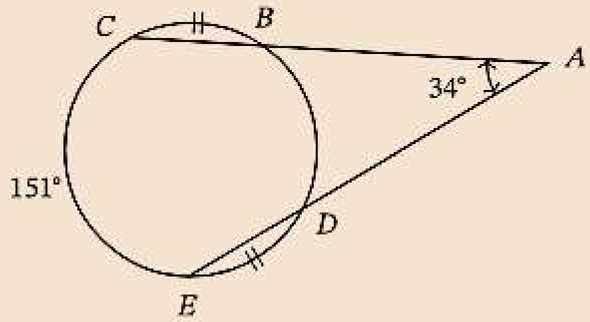
Q1: Find the value of z.



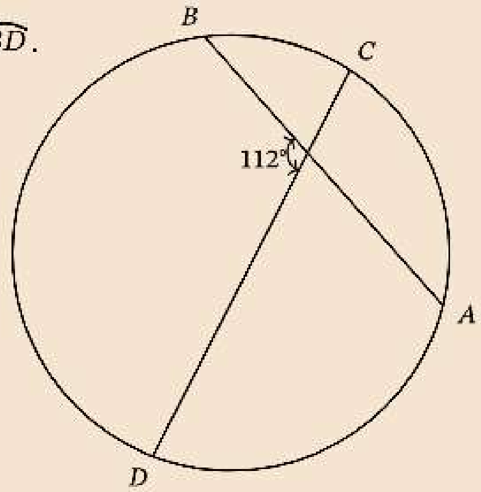
Q2: In the given figure, $m\widehat{CE} - m\widehat{BD} = 104°$. Determine $m\angle A$.



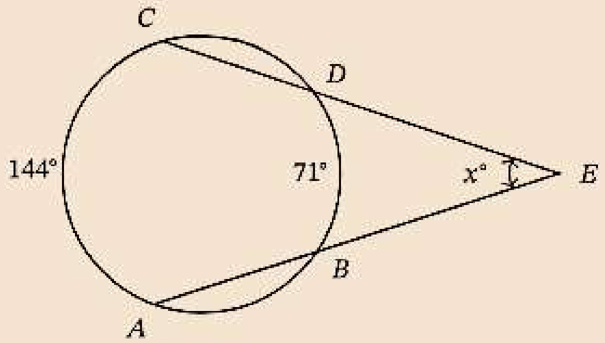
Q3: Determine $m\widehat{CB}$.



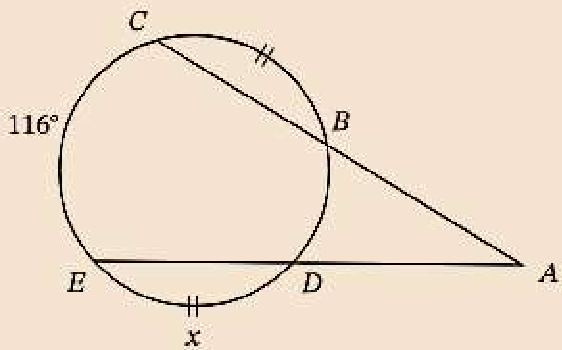
Q4: In the given figure, find $m\widehat{AC} + m\widehat{BD}$.



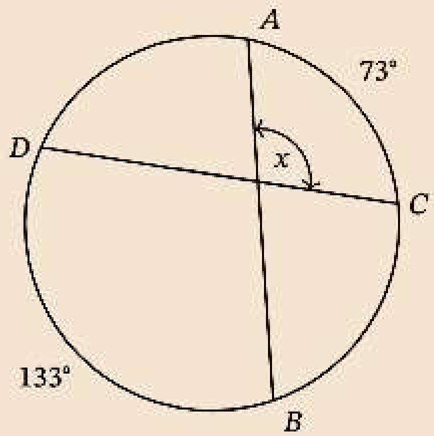
Q5: Find the value of x.



Q6: Given that $m\angle A = 31°$, find x.



Q7: Find x.



Applications of Proportion in the Circle

3 - 3

1 Determine the position of each of the following points with respect to the circle M, of radius length 10cm, then calculate the distance between each point from the centre of the circle.

A $P_M(A) = -36$ **B** $P_M(B) = 96$ **C** $P_M(C) = \text{zero}$

....

....

2 Find the power of the given point with respect to the circle M which its radius length is r:

A The point A where AM = 12cm and r = 9cm ....

B The point B where BM= 8 cm and r = 15 cm ....

C The point C where CM= 7 cm and r = 7 cm ....

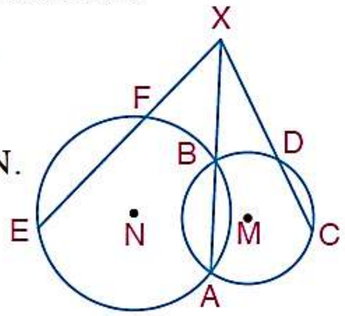
D The point D where DM= $\sqrt{17}$ cm and r = 4 cm ....

3 If the distance between a point and the centre of a circle equals 25cm and the power of this point with respect to the circle equals 400. Find the radius length of this circle. ....

4 The radius length of circle M equals 20cm, A is a point distant 16cm from the centre of the circle, the chord $\overline{BC}$ is drawn where $A \in \overline{BC}$ and $AB = 2\,AC$. Calculate the length of the chord $\overline{BC}$. ....

....

5 **In the figure opposite:** the two circles M and N are intersected at A and B where $\overrightarrow{AB} \cap \overrightarrow{CD} \cap \overrightarrow{EF} = \{X\}$, $XD = 2\,DC$, $EF = 10$cm and $P_N(X) = 144$.



A Prove that $\overleftrightarrow{AB}$ is a principle axis to the two circles M and N.

B Find the length of $\overline{XC}$ and $\overline{XF}$

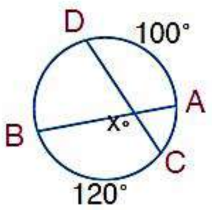
C Prove that CDFE is a cyclic quadrilateral.

....

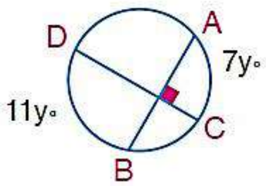
....

6 Use the given data of each figure to find the value of the symbol used in measurement.

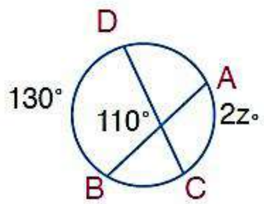
A



B



C



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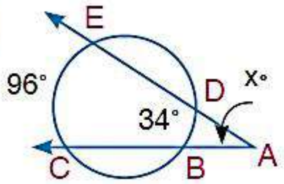
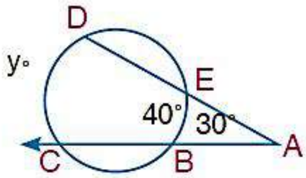
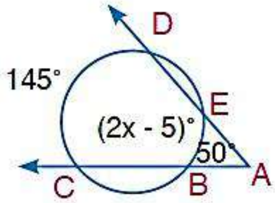
D



E

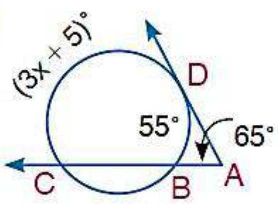


F

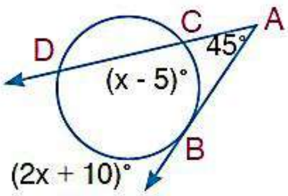


....

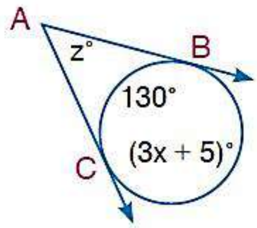
G



H

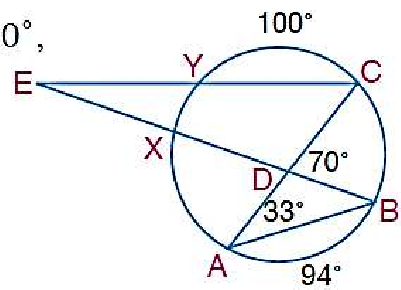


I



....

7 **In the figure opposite**: $m(\angle BAC) = 33°$, $m(\angle BDC) = 70°$, $m(\overset{\frown}{AB}) = 94°$, $X(\overset{\frown}{CY}) = 100°$ Find the measure of each of:



A $\overset{\frown}{XY}$

B $\overset{\frown}{AX}$

C $\angle BEC$

Unit Test

1 complete each of the following:

A The bisectors interior and the exterior of an angle of a triangle are ....

B The bisectors of the angles of a triangle intersect at ....

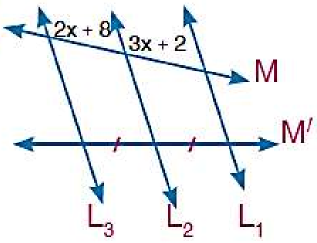
C If a line is drawn parallel to one side of a triangle and intersects the other two sides, then it ....

D The exterior bisector of the vertex angle of an isosceles triangle is .... the base of the triangle.

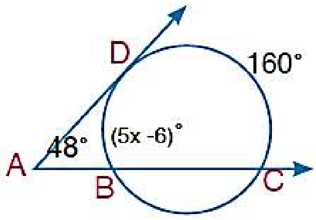
E If the power of a point A with respect to the circle M is a negative quantity, then A lies ....

2 Use the given in the following figures to find the value of the symbol used in measure.

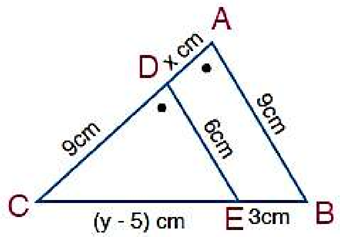
A



B

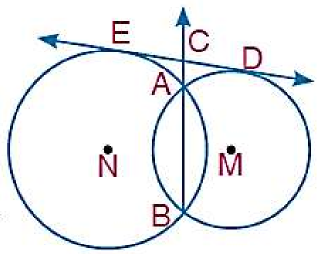


C



3 M and N are two intersecting circles at A and B.

$\overleftrightarrow{ED}$ is a common tangent to the two circles M and N at D and E respectively. $\overrightarrow{BA} \cap \overleftrightarrow{DE} = \{C\}$



A Prove that $\overleftrightarrow{BC}$ is a principle axis of the two circles.

B If A B = 9cm and $P_N(C) = 36$, Find the length of $\overline{CA}$ and $\overline{CD}$

Trigonometry

Futures Language Schools
Mathematical Department

First Secondary

First Term

Student Name:

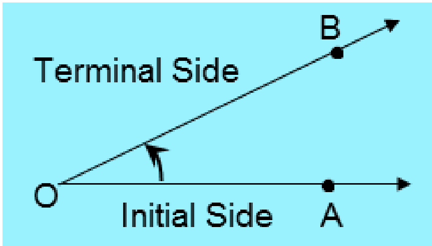
Lesson (1) Directed Angle

Lesson objectives

Related Links

- Identify Concept of directed angle.
- Identify Standard position of directed angle.
- Identify Positive and negative measure of the directed angle.
- Express Position of the directed angle in the coordinates plane.

The measure of the directed angle AOB ≠ the measure of the directed angle BOA.



Positive and negative measures of the directed angle

❶ The measure of the directed angle is positive if the direction of the arrow from the initial side to the terminal side is anticlockwise rotation.

❷ The measure of the directed angle is negative if the direction of the arrows is a clockwise rotation.

Remarks

❶ Each directed angle in its standard position has two measures. One of them is positive, the other one is negative, the sum of their absolute values = 360°

❷ If x° is the positive measure, then its negative measure = $x^\circ - 360^\circ$

❸ If $(-x^\circ)$ is the positive measure, then its negative measure = $360^\circ - x^\circ$.

Co-terminal angles: If θ° is the measure of the directed angle in its standard position, then the co-terminal angles can be fined by:

$$\theta^\circ = \theta^\circ \pm n \times 360^\circ$$

Example ❶

Determine the quadrant in which each of the following angles lies:

❶ 48º ❷ 217º ❸ 135º ❹ 295º ❺ 270º

Solution

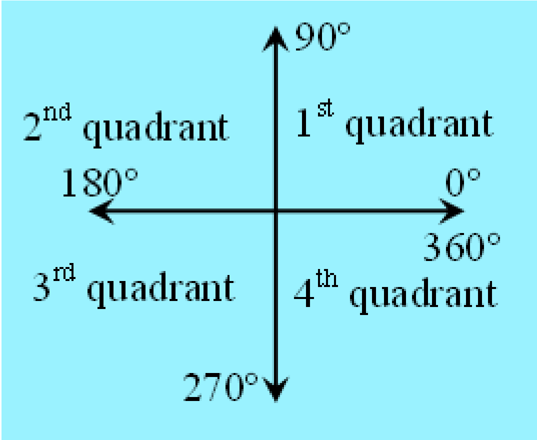
❶ 48º lies in 1^{st} quadrant

❷ 217º lies in 3^{rd} quadrant

❸ 135º lies in 2^{nd} quadrant

❹ 295º lies in 4^{th} quadrant

❺ 270º lies in y-axis



Example ❷

Determine the negative measure of the angles whose measures as follows:

❶ 32º ❷ 270º ❸ 56º ❹ 210º ❺ 315º

Solution

❶ 32º The negative measure = 32º – 360º = – 328º [32º ≅ – 328º]

❷ 270º The negative measure = 270º – 360º = – 90º [270º ≅ – 90º]

❸ 56º The negative measure = 56º – 360º = – 304º [56º ≅ – 304º]

❹ 210º The negative measure = 210º – 360º = – 150º [210º ≅ – 150º]

❺ 315º The negative measure = 315º – 360º = – 45º [315º ≅ – 45º]

Example ❸

Determine the positive measure of each of the following angles:

❶ -46° ❷ -246° ❸ -186° ❹ -300° ❺ -97°

Solution

❶ -46° The positive measure = $-46^\circ + 360^\circ = 314^\circ$ [$-46^\circ \cong 314^\circ$]

❷ -246° The positive measure = $-246^\circ + 360^\circ = 114^\circ$ [$-246^\circ \cong 114^\circ$]

❸ -186° The positive measure = $-186^\circ + 360^\circ = 174^\circ$ [$-186^\circ \cong 174^\circ$]

❹ -300° The positive measure = $-300^\circ + 360^\circ = 60^\circ$ [$-300^\circ \cong 60^\circ$]

❺ -97° The positive measure = $-97^\circ + 360^\circ = 263^\circ$ [$-97^\circ \cong 263^\circ$]

Example ❹

Determine the quadrant in which each of the following angles lies:

❶ 530° ❷ 1652° ❸ -740°

Solution

❶ 530°

$\because$ 530° equivalent to 170°

$\because$ 170° lies in 2^{nd} quadrant

$\therefore$ 530° lies in 2^{nd} quadrant

❷ 1652°

$\because$ 1652° equivalent to 212°

$\because$ 212° lies in 3^{rd} quadrant

$\therefore$ 1652° lies in 3^{rd} quadrant

❸ -740°

$\because$ -740° equivalent to 340°

$\because$ 340° lies in 4^{th} quadrant

$\therefore$ -740° lies in 4^{th} quadrant

PRACTICE

Q1: Find the first negative angle which is coterminal with 190°.

Q2: Find an angle with a positive measure and an angle with a negative measure that are coterminal with an angle of 332°.

A $512^\circ, -28^\circ$

B $602^\circ, -28^\circ$

C $28^\circ, -332^\circ$

D $692^\circ, -28^\circ$

E $28^\circ, -692^\circ$

Q3: Find one angle with positive measure and one angle with negative measure which are coterminal to an angle with measure $\frac{2\pi}{3}$. Given that: $\pi = 180^0$

A $\frac{8\pi}{3}, \frac{4\pi}{3}$

B $\frac{5\pi}{3}, -\frac{\pi}{3}$

C $\frac{4\pi}{3}, -\frac{4\pi}{3}$

D $\frac{8\pi}{3}, -\frac{4\pi}{3}$

E $-\frac{8\pi}{3}, \frac{4\pi}{3}$

Q4: Given the angle $\frac{273\pi}{3}$, find the principal angle. Given that: $\pi = 180^0$

A 0

B $\frac{\pi}{3}$

C 2π

D $\frac{4\pi}{3}$

E π

Q5: Given the angle $-\frac{23\pi}{5}$, find the principal angle. Given that: $\pi = 180^{0}$

A $\frac{5\pi}{3}$

B $-\frac{3\pi}{5}$

C $\frac{7\pi}{5}$

D $\frac{3\pi}{5}$

E $\frac{5\pi}{7}$

Q6: Find the smallest positive equivalent of $788°$.

Q7: Find the positive value of an angle coterminal with $25°$.

Q8: Find one angle with positive measure and one angle with negative measure which are coterminal to an angle with measure $\frac{3\pi}{2}$. Given that: $\pi = 180^{0}$

A $\frac{7\pi}{2}, \frac{\pi}{2}$

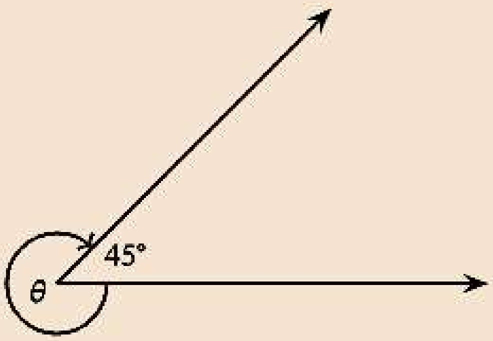
B $\frac{5\pi}{2}, -\frac{\pi}{2}$

C $\frac{\pi}{2}, -\frac{\pi}{2}$

D $\frac{7\pi}{2}, -\frac{\pi}{2}$

E $-\frac{7\pi}{2}, \frac{\pi}{2}$

Q9: Find the measure of $\angle\theta$.



Q10: In which quadrant does the angle -242° lie?

A. third quadrant

B. fourth quadrant

C. first quadrant

D. second quadrant

Q11: Which of the following angles is not equivalent to an angle measuring 25° in standard position?

A. -695°

B. -335°

C. 205°

D. 385°

Q12: Find a positive and a negative coterminal angle for 340°.

A. 10° and -690°

B. 690° and 10°

C. 20° and -700°

D. 700° and -20°

E. 700° and -700°

4 - 1 Directed Angle

1 Complete:

A) A directed angle is in the standard position if ....

B) It is said that the directed angles in the standard position are equivalent if ....

C) A directed angle is positive , if the rotation of the angle .... and is negative, if the rotation of the angle ....

D) If the terminal side of the directed angle lies on one of the coordinate axes, then it is called ....

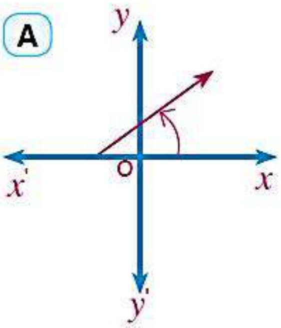
E) If (θ) is the measure of a directed angle in the standard position and $n \in Z$, then ($\theta + n \times 360^\circ$) is called .... angles.

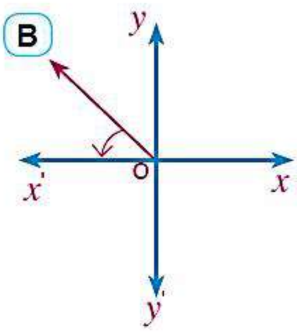
F) The smallest positive measure of the angle whose measure 530° is ....

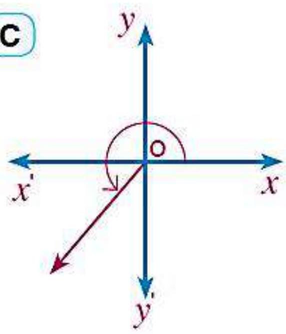
G) The angle whose measure 930° lies in the .... quadrant.

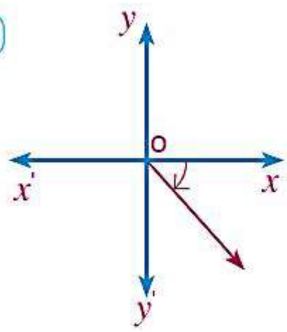
H) The smallest positive measure of the angle whose measure -690° is ....

2 Which of the following directed angles is in the standard position ....

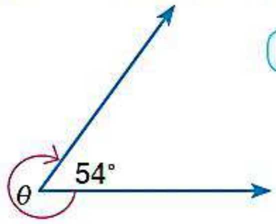
A)


B)


C)


D)


3 Find the measure of the directed angle θ in each of the following figures:

A)


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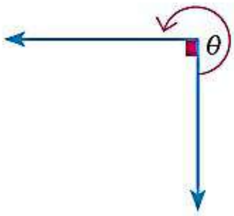
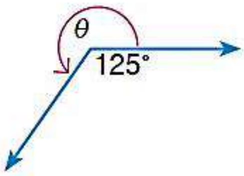
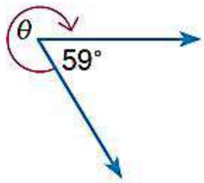


C)


D)


(4) Determine the quadrant in which each of the following angles lies on:

(A) 24° (B) 215° (C) -40° (D) -220° (E) 640°

....

(5) Show by drawing each of the following angles in the standard position:

(A) 32° (B) 140° (C) -80° (D) -110° (E) -315°

(6) Determine a negative measure for each of the following angles:

(A) 83° (B) 136° (C) 90°

....

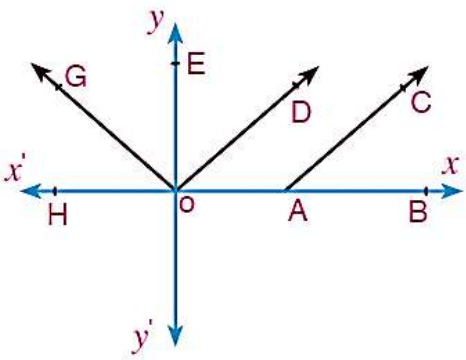
(D) 264° (E) 964° (F) 1070°

....

(7) Determine the smallest positive measure of each of the following angles:

(A) -183° (B) -217° (C) -315° (D) -570°

(8) **In the figure opposite:** which of the directed angles in the following ordered pairs is in the standard position? why?



(A) $(\overrightarrow{OA}, \overrightarrow{OD})$ (B) $(\overrightarrow{OG}, \overrightarrow{OC})$

(C) $(\overrightarrow{AB}, \overrightarrow{AC})$ (D) $(\overrightarrow{OE}, \overrightarrow{OD})$

(E) $(\overrightarrow{OD}, \overrightarrow{OG})$ (F) $(\overrightarrow{OB}, \overrightarrow{OG})$

....

Lesson (2) Systems of Measuring Angle

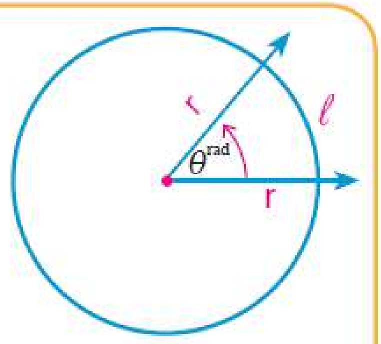
Lesson objectives

Related Links

- Identify Concept of radian measure of an angle.
- Classify Relation between radian and degree measure.
- Find the length of an arc in a circle.

Definition

If θ^{rad} is the radian measure of the central angle in a circle of radian r subtends an arc of length ℓ, then length of the arc equals the product of the radian measure of the central angle and the radius of its circle:



$$\boxed{\ell = \theta^{rad} \times r}$$

Systems for measuring

First: Degree measure system

$1^{\circ} = 60'$, $1' = 60''$

Second: Radian measure system

The radian measure of a central angle in a circle

$$= \frac{\text{The length of the arc subtended by the angle}}{\text{The radius length of this circle}}$$

Important Rules

❶ $\theta^{rad} = \frac{L}{r}$ ❷ $L = \theta^{rad} \times r$ ❸ $r = \frac{L}{\theta^{rad}}$

Relation between the degree measure and the radian

- $\theta^{rad} \longrightarrow \pi$
- $\theta^{\circ} \longrightarrow 180^{\circ}$

Important Rules

❶ $\frac{\theta^{\circ}}{180^{\circ}} = \frac{\theta^{rad}}{\pi}$ ❷ $\theta^{rad} = \theta^{\circ} \times \frac{\pi}{180^{\circ}}$ ❸ $\theta^{\circ} = \theta^{rad} \times \frac{180^{\circ}}{\pi}$

Example ❶

Find the degree measure (in degree, minute and second) of each of the following angles whose radian measures is 2.27^{rad}

Solution

$2.27^{rad} = 2.27^{rad} \times 180^{\circ} \div \pi = 130.06^{\circ} = 130^{\circ}\, 3'\, 41''$

Example ❷

Find the degree measure and radian measure of the central angle that subtends an arc of length L in a circle of radius length r if L = 12 cm., r = 10 cm.

Solution

L = 12 cm. , r = 10 cm.

$\theta^{rad} = \frac{\ell}{r} = \frac{12}{10} = 1.2^{rad}$

$\theta^{\circ} = 1.2^{rad} \times 180^{\circ} \div \pi = 68.75^{\circ} = 68^{\circ}\, 45'\, 18''$

Example ❸

Find the radius length of the circle if the measure of the central angle drawn in it is θ and the length of subtends arc (L) if: $\theta = 0.767^{rad}$, L = 38.35 cm.

Solution

$\theta = 0.767^{rad}$, $L = 38.35$ cm.

$r = \frac{\ell}{\theta^{rad}} = \frac{38.35}{0.767^{rad}} = 50$ cm

Example ❹

Find the length of the arc in a circle with radius length r to nearest one decimal digit of cm. if it is opposite to a central angle with measure θ if:
$r = 20$ cm., $\theta = 2.43^{rad}$

Solution

$r = 20$ cm., $\theta = 2.43^{rad}$

$\ell = \theta^{rad} \times r = 2.43^{rad} \times 20 = 48.6$ cm

Example ❺

Geometry: the radius length of a circle equals 4 cm. The inscribed angle $\angle ABC$ of measure 30° is drawn in it. Find the length of the smaller arc $\overset{\frown}{AC}$

Solution

$\therefore \theta° = 60°$

[measure of the central angle = 2 measure of the inscribed angle]

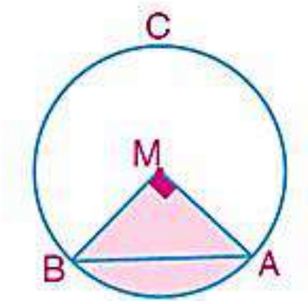
$\theta^{rad} = 60° \times \pi \div 180° = \frac{1}{3}\pi$

$r = \frac{\ell}{\theta^{rad}} = \frac{4}{\frac{1}{3}\pi} = \frac{12}{\pi}$ cm

$\because$ **Circumference of the circle** $= 2\pi r = 2 \times \pi \times \frac{12}{\pi} = 24$ cm

Example ❻

Geometry: In the figure opposite: if the area of the right angled triangle M A B at M equals 32 cm², then find the perimeter of the coloured figure to the nearest hundredth. ....



Solution

$\because$ **Area of** Δ **MAB = 32**

$\therefore \frac{1}{2}$ *Base* $\times$ *height* $= 32$

$\therefore AM \times BM = 64$

$\therefore r^2 = 64$

$\therefore r = 8$ cm

$\theta^{rad} = 90° \times \pi \div 180° = \frac{1}{2}\pi$

$\therefore \ell(\widehat{AB}) = \theta^{rad} \times r = \frac{1}{2}\pi \times 8 = 4\pi$ cm

$\therefore$ **Perimeter of shaded part = 8 + 8 + 4π = 28.57 cm**

PRACTICE (1)

Q1: Convert $\frac{\pi}{3}$ to degrees.

Q2: Convert 0.5 rad to degrees giving the answer to the nearest second.

A 90°

B $0^\circ 9' 33''$

C $0^\circ 31''$

D $28^\circ 38' 52''$

E $57^\circ 17' 45''$

Q3: Convert -3.3 rad to degrees giving the answer to the nearest second.

A 126°

B $170^\circ 55' 26''$

C $359^\circ 56' 33''$

D $358^\circ 56' 58''$

E $341^\circ 50' 52''$

Q4: A gymnast circles a pommel horse by an angle of 50°. Find the angle in radians giving the answer to one decimal place.

Q5: Find the values of two angles in degrees given their sum is 74° and their difference is $\frac{\pi}{6}$. Give the answer to the nearest degree.

A $52^\circ, 22^\circ$

B $62^\circ, 22^\circ$

C $62^\circ, 12^\circ$

D $62^\circ, 32^\circ$

Q6: Find the values of two angles in radians given their sum is $74°$ and their difference is $\frac{2\pi}{9}$. Give the answer to two decimal places.

A 0.3 rad, 1.17 rad

B 0.47 rad, 0.99 rad

C 0.3 rad, 0.99 rad

D 0.47 rad, 1.17 rad

Q7: Convert $142°46'48''$ to radians giving the answer to three decimal places.

Q8: The shadow of a sundial changes at a rate of $15°$ every hour. Find the angle of the shadow's position 5 hours after sunrise giving the answer in radians to three decimal places.

Q9: Convert $360°$ to radians giving the answer in terms of π.

A $\frac{360}{\pi}$

B $\frac{\pi}{2}$

C π

D 2π

E 360π

Q10: What is the equivalent measure of $360°$ in radians?

A 2π

B $\frac{\pi}{2}$

C π

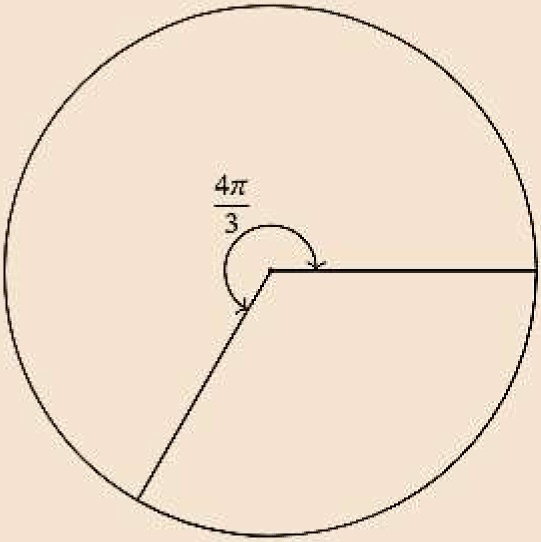
D 3π

E 4π

PRACTICE (2)

Q1: Find the length of the blue arc given the radius of the circle is 8 cm.

Give the answer to one decimal place.



Q2: An arc has a measure of $\frac{2\pi}{3}$ radians and a radius of 9. Work out the length of the arc, giving your answer in terms of π, in its simplest form.

A 9π

B 3π

C 2π

D 18π

E 6π

Q3: An arc has a measure of $\frac{\pi}{8}$ radians and a radius of 6. Work out the length of the arc, giving your answer in terms of π, in its simplest form.

A $\frac{3\pi}{8}$

B $\frac{3\pi}{2}$

C $\frac{3\pi}{4}$

D $\frac{4\pi}{3}$

E $\frac{2\pi}{3}$

4 - 2 Degree measure and radian measure of an angle

First: Multiple choice:

1. The angle of measure 60° in the standard position is equivalent to the angle of measure:

   (A) 120° (B) 240° (C) 300° (D) 420°

2. The angle of measure $\frac{31\pi}{6}$ lies in the .... quadrant

   (A) First (B) second (C) third (D) fourth

3. The angle of measure $\frac{-9\pi}{4}$ lies in the .... quadrant

   (A) First (B) second (C) third (D) fourth

4. If the sum of measures of the interior angles of a regular polygon equals 180° $(n - 2)$ where n is the number of its sides, then the measure of the angle of a regular pentagon in radian measure equals: ....

   (A) $\frac{\pi}{3}$ (B) $\frac{7\pi}{2}$ (C) $\frac{3\pi}{5}$ (D) $\frac{2\pi}{3}$

5. The angle of measure $\frac{7\pi}{3}$ its degree measure equals ....

   (A) 105° (B) 210° (C) 420° (D) 840°

6. If the degree measure of an angle is $64^\circ\ 48'$, then its radian measure equals ....

   (A) 0.18^{rad} (B) 0.36^{rad} (C) $0.18\ \pi$ (D) $0.36\ \pi$

7. The arc length in a circle of diameter length 24 cm and opposite to a central angle of measure 30° is ....

   (A) 2π cm (B) 3π cm (C) 4π cm (D) 5π cm

8. The measure of the central angle in a circle of radius length 15 cm and opposite to an arc length 5πcm equals ....

   (A) 30° (B) 60° (C) 90° (D) 180°

9) If the measure of an angle of a triangle equals 75° and the measure of another angle equals $\frac{\pi}{4}$, then the radian measure of the third angle equals ....

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{5\pi}{12}$

Second: Answer the following questions:

10) In terms of π, find the radian measure of the following angles

A) 225° .... B) 240° ....

C) -135° .... D) 300° ....

E) 390° .... F) 780° ....

11) Find the radian measure of the following angles approximating the result to the nearest three decimal places:

A) 56.6° B) $25^\circ\ 18'$ C) $160^\circ\ 50'\ 48''$

....

12) Find the dgree measure of the following angles approximating the result to the nearest second:

A) 0.49^{rad} B) 2.27^{rad} C) $-3\frac{1}{2}^{rad}$

....

13) θ is a central angle in a circle of radius r and subtends an arc of length L :

A) If $r = 20$ cm and $\theta = 78^\circ\ 15'\ 20''$ then find L. ....(to the nearest tenth)

B) If $L = 27.3$ cm and $\theta = 78^\circ\ 0'\ 24''$ then find r. ....(to the nearest tenth)

14) A central angle of measure 150° and subtends an arc length 11cm. Calculate its radius length (to the nearest tenth).

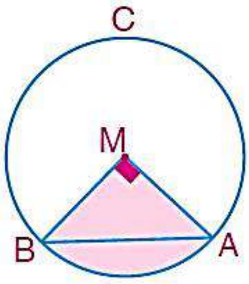
....

15) Find the radian and degree measure of the central angle which subtends an arc length 8.7cm in a circle of radius length 4cm..........

(16) **Geometry:** the measure of an angle of a triangle is 60° and the measure of another angle is $\frac{\pi}{4}$. Find the radin measure and the degree measure of the third angle. ....

(17) **Geometry:** the radius length of a circle equals 4 cm. The inscribed angle $\angle ABC$ of measure 30° is drawn in it. Find the length of the smaller arc $\overset{\frown}{AC}$ ....

(18) **Geometry:** In the figure opposite: if the area of the right angled triangle M A B at M equals 32 cm^2 , then find the perimeter of the coloured figure to the nearest hundredth. ....



(19) **Geometry:** the diameter length in a circle equals 24cm and the chord $\overline{AC}$ is drawn such that m $(\angle BAC) = 50^\circ$. Find the length of the smaller arc $\overset{\frown}{AC}$ approximating the result to the nearest hundredth. ....

(20) **Distances:** What is the distance covered by the point on the end of the minute hand in 10 minutes, if the hand length is 6cm?

....

(21) **Astronomy:** A satellite revolves around the Earth in a circular path way a full revolution every 6 hours. If the radius length of its path way equals 9000km, then find its speed in km/h ....

Lesson (3) Trigonometric functions

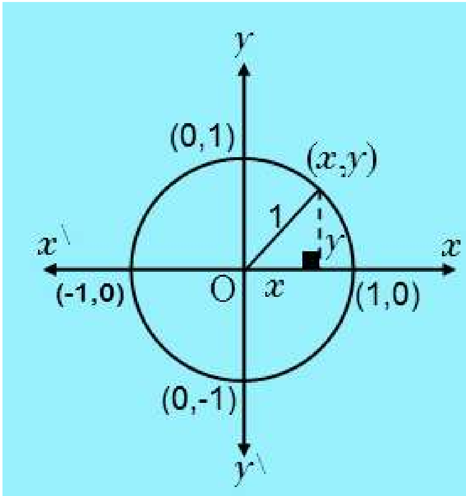
Related Links

Lesson objectives

- Identify Unit circle.
- Define Basic trigonometric functions.
- Find Reciprocals of basic trigonometric functions.
- Recognize Signs of the trigonometric functions.

The unit circle

$x^2 + y^2 = 1$



The basic trigonometric functions

$\text{Cos } \theta = x$

$\text{Sin } \theta = y$

$\text{Tan } \theta = \dfrac{y}{x} = \dfrac{\text{Sin } \theta}{\text{Cos } \theta}$

The co-terminal angles have the same trigonometric functions

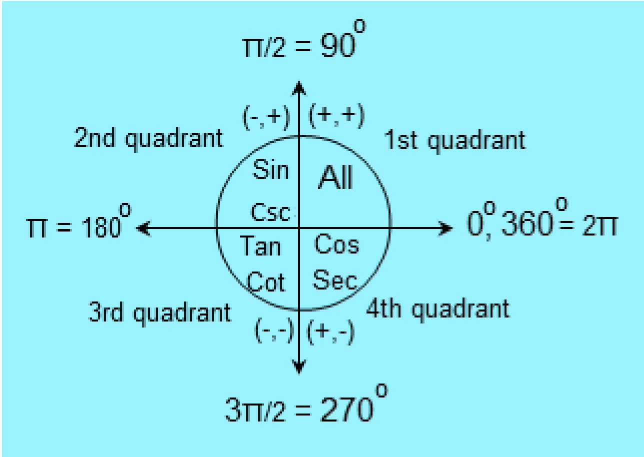
- $\text{Cos } (\theta + 2n\pi) = \text{Cos } \theta = x$
- $\text{Sin } (\theta + 2n\pi) = \text{Sin } \theta = y$
- $\text{Tan } (\theta + 2n\pi) = \text{Tan } \theta = \dfrac{y}{x}$

Definition of the reciprocals of the basic trigonometric functions

$\text{Sin } \theta = \dfrac{1}{\text{Csc } \theta}$ $\quad$ $\text{Cos } \theta = \dfrac{1}{\text{Sec } \theta}$ $\quad$ $\text{Tan } \theta = \dfrac{1}{\text{Cot } \theta}$

$\text{Csc } = \dfrac{1}{\text{Sin } \theta}$ $\quad$ $\text{Sec } \theta = \dfrac{1}{\text{Cos } \theta}$ $\quad$ $\text{Cot } \theta = \dfrac{1}{\text{Tan } \theta}$

The signs of the trigonometric functions



Example ❶

Determine the signs of the following trigonometric functions:

❶ tan 320° ❷ sin 160° ❸ sec 750° ❹ csc $\frac{4\pi}{5}$ ❺ cos (–200°)

Solution

❶ tan 320° $\because$ 320° lies in 4^{th} quadrant

$\therefore$ tan 320° is negative

❷ sin 160° $\because$ 160° lies in 2^{nd} quadrant

$\therefore$ sin 160° is positive

❸ sec 750° $\because$ 750° = 30° lies in 1^{st} quadrant

$\therefore$ sec 750° is positive

❹ csc $\frac{4\pi}{5}$ $\because \frac{4\pi}{5} = \frac{4\pi}{5} \times 180° \div \pi = 144°$ lies in 2^{nd} quadrant

$\therefore$ csc $\frac{4\pi}{5}$ is positive

❺ cos (–200°) $\because$ –200° = –200° + 360° = 160° lies in 2^{nd} quadrant

$\therefore$ cos (–200°) is negative

Example ❷

If θ is the measure of a directed angle in its standard position, B is the point of the intersection of its terminal side with the unit circle, find all trigonometric functions of θ in each of the following cases:

① $\left(-\frac{3}{5}, \frac{4}{5}\right)$ ② $\left(\frac{5}{13}, \frac{12}{13}\right)$ ③ $(0.6, y), y > 0$

Solution

① $\left(-\frac{3}{5}, \frac{4}{5}\right)$ ∵ x is –ve & y is +ve ∴ $\left(-\frac{3}{5}, \frac{4}{5}\right)$ lies in 2^{nd} quadrant

Sin θ $= \frac{4}{5}$	Cos θ $= -\frac{3}{5}$	Tan θ $= -\frac{4}{3}$
Csc θ $= \frac{5}{4}$	Sec θ $= -\frac{5}{3}$	Cot θ $= -\frac{3}{4}$

② $\left(\frac{5}{13}, \frac{12}{13}\right)$ ∵ x is +ve & y is +ve ∴ $\left(\frac{5}{13}, \frac{12}{13}\right)$ lies in 1^{st} quadrant

Sin θ $= \frac{12}{13}$	Cos θ $= \frac{5}{13}$	Tan θ $= \frac{12}{5}$
Csc θ $= \frac{13}{12}$	Sec θ $= \frac{13}{5}$	Cot θ $= \frac{5}{12}$

③ $(0.6, y), y > 0$ ∵ $x^2 + y^2 = 1$ ∴ $(0.6)^2 + y^2 = 1$

∴ $y^2 = 1 - 0.36$ ∴ $y^2 = 0.64$ ∴ $y = \pm\sqrt{0.64} = 0.8 \quad (y > 0)$

∴ $\left(\frac{3}{5}, \frac{4}{5}\right)$ ∵ x is +ve & y is +ve ∴ $\left(\frac{3}{5}, \frac{4}{5}\right)$ lies in 1^{st} quadrant

Sin θ $= \frac{4}{5}$	Cos θ $= \frac{3}{5}$	Tan θ $= \frac{4}{3}$
Csc θ $= \frac{5}{4}$	Sec θ $= \frac{5}{3}$	Cot θ $= \frac{3}{4}$

Example ❸

If $\theta \in \left]\frac{3\pi}{2}, 2\pi\right[$ and $\cos\theta = \frac{5}{13}$, find all trigonometric functions of θ

Solution

$\because \cos\theta = \frac{5}{13}$ $\quad \therefore x = \frac{5}{13}$ $\quad \because x^2 + y^2 = 1$

$\therefore \left(\frac{5}{13}\right)^2 + y^2 = 1$ $\quad \therefore y^2 = 1 - \frac{25}{169}$ $\quad \therefore y^2 = \frac{144}{169}$

$\therefore y = \pm\sqrt{\frac{144}{169}} = -\frac{12}{13}$ $\quad \boxed{\theta \in \left]\frac{3\pi}{2}, 2\pi\right[}$

$\sin\theta = -\frac{12}{13}$	$\cos\theta = \frac{5}{13}$	$\tan\theta = -\frac{12}{5}$
$\csc\theta = -\frac{13}{12}$	$\sec\theta = \frac{13}{5}$	$\cot\theta = -\frac{5}{12}$

Special Angles

θ	Sin θ	Cos θ	Tan θ
0° or 360°	0	1	0
$90^\circ = \frac{\pi}{2}$	1	0	*undefined*
$180^\circ = \pi$	0	-1	0
$270^\circ = \frac{3\pi}{2}$	-1	0	*undefined*
$30^\circ = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$60^\circ = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$45^\circ = \frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1

Example ❹

Find the value of:

4 sin 30º sin 90º – cos 0º sec 60º + 5 tan 45º + 10 cos² 45º sin 270º – tan 30º sin 180º

Solution

4 sin 30º sin 90º – cos 0º sec 60º + 5 tan 45º + 10 cos² 45º sin 270º – tan 30º sin 180º

$$= 4 \times \frac{1}{2} \times 1 - 1 \times 2 + 5 \times 1 + 10 \times \left(\frac{\sqrt{2}}{2}\right)^2 \times -1 - \frac{\sqrt{3}}{3} \times 0 = 0$$

Example ❺

Prove that:

$$\sin^2 60^\circ + \sin^2 45^\circ + \sin^2 30^\circ = \cos^2 \frac{\pi}{6} \sin \frac{\pi}{2} - \frac{1}{3} \tan^2 \frac{\pi}{3} \cos \pi + \cos^2 \frac{\pi}{3} \sin \frac{3\pi}{2}$$

Solution

L.H.S. $= \sin^2 60^\circ + \sin^2 45^\circ + \sin^2 30^\circ$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{2} \quad \rightarrow ①$$

R.H.S. $= \cos^2 \frac{\pi}{6} \sin \frac{\pi}{2} - \frac{1}{3} \tan^2 \frac{\pi}{3} \cos \pi + \cos^2 \frac{\pi}{3} \sin \frac{3\pi}{2}$

Note that: $\pi = 180^\circ$

$$= \cos^2 30^\circ \sin 90^\circ - \frac{1}{3} \tan^2 60^\circ \cos 180^\circ + \cos^2 60^\circ \sin 270^\circ$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 \times 1 - \frac{1}{3} \times \left(\sqrt{3}\right)^2 \times -1 + \left(\frac{1}{2}\right)^2 \times -1 = \frac{3}{2} \quad \rightarrow ②$$

PRACTICE (1)

Q1: Find $\sin\theta$, given θ is in standard position and its terminal side passes through the point $\left(\frac{3}{5}, -\frac{4}{5}\right)$

A $\frac{4}{5}$

B $\frac{3}{5}$

C $-\frac{3}{5}$

D $-\frac{4}{5}$

Q2: Find $\tan\theta$ given θ is in standard position and its terminal side passes through the point $\left(-\frac{3}{5}, -\frac{4}{5}\right)$.

A $-\frac{3}{4}$

B $-\frac{4}{3}$

C $\frac{3}{4}$

D $\frac{4}{3}$

Q3: Find $\sec\theta$, given θ is in standard position and its terminal side passes through the point $\left(\frac{4}{5}, \frac{3}{5}\right)$.

A $\frac{5}{3}$

B $\frac{5}{4}$

C $\frac{4}{5}$

D $\frac{3}{5}$

PRACTICE (2)

Q1: The terminal side of θ in standard position intersects with the unit circle at the point B with coordinates $\left(\frac{8}{17}, \frac{15}{17}\right)$. Find $\sec\theta$.

A $\frac{17}{8}$

B $-\frac{17}{8}$

C $\frac{17}{15}$

D $\frac{15}{8}$

Q2: The terminal side of $\angle AOB$ in standard position intersects with the unit circle at the point B with coordinates $(-x, -x)$ where x is a postive number. Find $\sin\theta$.

A $-\frac{1}{\sqrt{2}}$

B 1

C $\frac{1}{\sqrt{2}}$

D -1

PRACTICE (3)

Q1: Determine the quadrant in which θ lies if $\cos\theta < 0$ and $\sin\theta < 0$.

A the third quadrant

B the second quadrant

C the fourth quadrant

D the first quadrant

Q2: Determine the quadrant in which θ lies if $\cos\theta > 0$ and $\sin\theta < 0$.

A the first quadrant

B the second quadrant

C the third quadrant

D the fourth quadrant

Q3: The angle θ is in standard position where $\sec\theta < 0$. In which quadrants does the terminal side of θ lie?

A first or third

B second or third

C first or fourth

D second or fourth

Q4: Determine the quadrant in which θ lies if $\cos\theta < 0$ and $\sin\theta > 0$.

A the first quadrant

B the fourth quadrant

C the second quadrant

D the third quadrant

Trigonometric Functions

4 - 3

First: Multiple Choice:

1. If θ is an angle in the standard position and its terminal side passes through the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, then $\sin\theta$ equals: ....

   (A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{2}{\sqrt{3}}$

2. If $\sin\theta = \frac{1}{2}$ where θ is an acute angle , then m ($\angle\theta$) equals ....

   (A) 30° (B) 45° (C) 60° (D) 90°

3. If $\sin\theta = -1$, $\cos\theta = 0$, then the measure of angle θ equals ....

   (A) $\frac{\pi}{2}$ (B) π (C) $\frac{3\pi}{2}$ (D) 2π

4. If $\csc\theta = 2$ where θ is the measure of an acute angle, then measure of angle θ equals ....

   (A) 15° (B) 30° (C) 45° (D) 60°

5. If $\cos\theta = \frac{1}{2}$, $\sin\theta = -\frac{\sqrt{3}}{2}$, then measure of angle θ equals ....

   (A) $\frac{2\pi}{3}$ (B) $\frac{5\pi}{6}$ (C) $\frac{5\pi}{3}$ (D) $\frac{11\pi}{6}$

6. If $\tan\theta = 1$ where θ is a positive acute angle, then measure of angle θ equals ....

   (A) 10° (B) 30° (C) 45° (D) 60°

7. $\tan 45^\circ + \cot 45^\circ - \sec 60^\circ$ equals ....

   (A) Zero (B) $\frac{1}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) 1

8. If $\cos\theta = \frac{\sqrt{3}}{2}$ where θ is an acute angle , then $\sin\theta$ equals ....

   (A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{2}{\sqrt{3}}$ (D) $\frac{\sqrt{3}}{2}$

Second: Answer the following questions:

9. Find all trigonometric functions of angle θ drawn in the standard position and its terminal side intersects the unit circle and passes through each of the following points.

A $(\frac{2}{3}, \frac{\sqrt{5}}{3})$ B $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ C $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ D $(-\frac{3}{5}, -\frac{4}{5})$

....

10. If θ is the measure of the directed angle in the standard position and its terminal side intersects the unit circle at the given point, find all trigonometric function of the angle θ in each of the following cases :

A $(3a, -4a)$ where $a > 0$ ....

B $(\frac{3}{2}a, -2a)$ where $\frac{3\pi}{2} < \theta < 2\pi$ ....

11. Determine the sign of each of the following trigonometric function:

A $\sin 240^\circ$ ....

B $\tan 365^\circ$ ....

C $\csc 410^\circ$ ....

D $\cot \frac{9\pi}{4}$ ....

E $\sec - \frac{9\pi}{4}$ ....

F $\tan \frac{-20\pi}{9}$ ....

12. **Find the value of each of the following:**

A $\cos \frac{\pi}{2} \times \cos 0 + \sin \frac{3\pi}{2} \times \sin \frac{\pi}{2}$ ....

B $\tan^2 30^\circ + 2 \sin^2 45^\circ + \cos^2 90^\circ$ ....

Lesson (4) Related Angles

Lesson objectives

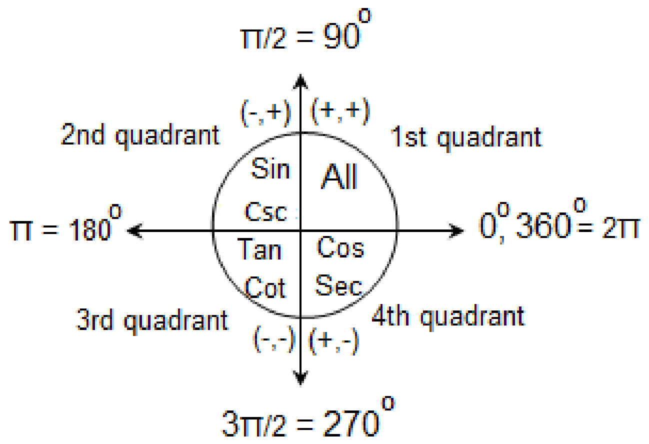
Related Links

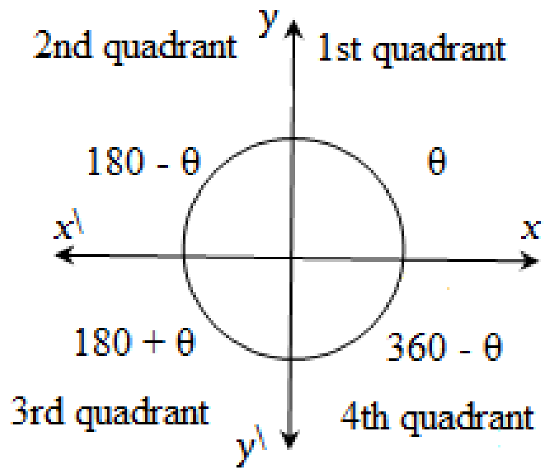
- Classify the relation between trigonometric functions of angles $(\theta, \theta \pm 90°)$, $(\theta, \theta \pm 180°)$ and $(\theta, \theta \pm 270°)$.
- Find the general solution of trigonometric equations in the form: sin x = cos y , csc x = sec y and tan x = cot y

Remember that:

$\sin(90° - \theta) = \cos\theta$	$\csc(90° - \theta) = \sec\theta$
$\cos(90° - \theta) = \sin\theta$	$\sec(90° - \theta) = \csc\theta$
$\tan(90° - \theta) = \cot\theta$	$\cot(90° - \theta) = \tan\theta$
$\sin(-\theta) = -\sin\theta$	$\csc(-\theta) = -\csc\theta$
$\cos(-\theta) = \cos\theta$	$\sec(-\theta) = \sec\theta$
$\tan(-\theta) = -\tan\theta$	$\cot(-\theta) = -\cot\theta$
$\sin(360° - \theta) = -\sin\theta$	$\csc(360° - \theta) = -\csc\theta$
$\cos(360° - \theta) = \cos\theta$	$\sec(360° - \theta) = \sec\theta$
$\tan(360° - \theta) = -\tan\theta$	$\cot(360° - \theta) = -\cot\theta$
$\sin(90° + \theta) = \cos\theta$	$\csc(90° + \theta) = \sec\theta$
$\cos(90° + \theta) = -\sin\theta$	$\sec(90° + \theta) = -\csc\theta$
$\tan(90° + \theta) = -\cot\theta$	$\cot(90° + \theta) = -\tan\theta$
$\sin(180° - \theta) = \sin\theta$	$\csc(180° - \theta) = \csc\theta$
$\cos(180° - \theta) = -\cos\theta$	$\sec(180° - \theta) = -\sec\theta$
$\tan(180° - \theta) = -\tan\theta$	$\cot(180° - \theta) = -\cot\theta$
$\sin(180 + \theta) = -\sin\theta$	$\csc(180 + \theta) = -\csc\theta$
$\cos(180 + \theta) = -\cos\theta$	$\sec(180 + \theta) = -\sec\theta$
$\tan(180 + \theta) = \tan\theta$	$\cot(180 + \theta) = \cot\theta$
$\sin(270 - \theta) = -\cos\theta$	$\csc(270 - \theta) = -\sec\theta$
$\cos(270 - \theta) = -\sin\theta$	$\sec(270 - \theta) = -\csc\theta$
$\tan(270 - \theta) = \cot\theta$	$\cot(270 - \theta) = \tan\theta$
$\sin(270 + \theta) = -\cos\theta$	$\csc(270 + \theta) = -\sec\theta$
$\cos(270 + \theta) = \sin\theta$	$\sec(270 + \theta) = \csc\theta$
$\tan(270 + \theta) = -\cot\theta$	$\cot(270 + \theta) = -\tan\theta$

An Important remark





Example ❶

Find the value of each of the following:

❶ $\sin 240^\circ$ ❷ $\cos \frac{5\pi}{3}$ ❸ $\cos 570^\circ$ ❹ $\tan \frac{14\pi}{3}$ ❺ $\tan(-150^\circ)$

Solution

❶ $\sin 240^\circ = \sin(180^\circ + 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$

❷ $\cos \frac{5\pi}{3} = \cos 300^\circ = \cos(360^\circ - 60^\circ) = \cos 60^\circ = \frac{1}{2}$

❸ $\cos 570^\circ = \cos 210^\circ = \cos(180^\circ + 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$

❹ $\tan \frac{14\pi}{3} = \tan 840^\circ = \tan 120^\circ = \tan(180^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3}$

❺ $\tan(-150^\circ) = \tan 210^\circ = \tan(180^\circ + 30^\circ) = \tan 30^\circ = \frac{\sqrt{3}}{3}$

Example ❷

If $\cos c = -\frac{4}{5}$ where $c \in \,]90^\circ, 180^\circ[$, find the value of each of the following:

❶ $\sin(180^\circ - c)$ **❷ $\sec(360^\circ - c)$**

❸ $\cos(-c)$ **❹ $\tan(c - 180^\circ)$**

❺ $\cos(180^\circ + c) - \cot(270^\circ + c)$

$x \longrightarrow \cos$

$y \longrightarrow \sin$

Solution

$\because x^2 + y^2 = 1$

$\therefore \left(\frac{-4}{5}\right)^2 + y^2 = 1$

$\therefore y^2 = 1 - \frac{16}{25} = \frac{9}{25}$

$\therefore y = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$ $\because c \in \,]90^\circ, 180^\circ[$

$\therefore$ c lies in 2nd quadrant $\therefore y = \frac{3}{5}$ $\therefore \sin c = \frac{3}{5}$

$\sin C = \frac{3}{5}$	$\cos C = -\frac{4}{5}$	$\tan C = \frac{4}{3}$
$\csc C = \frac{5}{3}$	$\sec C = -\frac{5}{4}$	$\cot C = \frac{3}{4}$

❶ $\sin(180^\circ - c) = \sin c = \frac{3}{5}$

❷ $\sec(360^\circ - c) = \sec c = -\frac{5}{4}$

❸ $\cos(-c) = \cos c = -\frac{4}{5}$

❹ $\tan(c - 180^\circ) = \tan(c - 180^\circ + 360^\circ) = \sin(180^\circ + c) = -\sin c = -\frac{3}{5}$

❺ $\cos(180^\circ + c) - \cot(270^\circ + c) = -\cos c - (-\tan c) = -\left(-\frac{4}{5}\right) + \frac{4}{3} = \frac{32}{15}$

Remarks

- If sin x = cos , then $x \pm y = 90° + 360°$ n
- If csc x = sec , then $x \pm y = 90° + 360°$ n
- If tan x = cot , then $x + y = 90° + 180°$ n
- If x & y are complementary (sum = 90º), then: sin x = cos , tan x = cot y and csc x = sec y

Example ❸

If sin (3 x + 28º) = cos (2 x – 13º) , find the values of x where 0º < x < 90º

Solution

$\because$ sin (3 x + 28º) = cos (2 x – 13º)

$\therefore$ (3 x + 28º) $\pm$ (2 x – 13º) = 90° + 360° n

$\therefore (3x + 28^\circ) + (2x - 13^\circ) = 90^\circ + 360^\circ n$	$\therefore (3x + 28^\circ) - (2x - 13^\circ) = 90^\circ + 360^\circ n$
$\therefore 3x + 28^\circ + 2x - 13^\circ = 90^\circ + 360^\circ n$	$\therefore 3x + 28^\circ - 2x + 13^\circ = 90^\circ + 360^\circ n$
$\therefore \mathbf{5x + 15^\circ = 90^\circ + 360^\circ n}$	$\therefore \mathbf{x + 41^\circ = 90^\circ + 360^\circ n}$
At n = 0	**At n = 0**
$\therefore 5x + 15^\circ = 90^\circ$	$\therefore x + 41^\circ = 90^\circ$
$\therefore 5x = 90^\circ - 15^\circ = 75^\circ$	$\therefore x = 90^\circ - 41^\circ$
$\therefore x = 15^\circ < 90^\circ$ $\surd$	$\therefore x = 49^\circ < 90^\circ$ $\surd$
At n = 1	**At n = 1**
$\therefore 5x + 15^\circ = 90^\circ + 360^\circ$	$\therefore x + 41^\circ = 90^\circ + 360^\circ$
$\therefore 5x = 450^\circ - 15^\circ = 435^\circ$	$\therefore x = 450^\circ - 41^\circ$
$\therefore x = 87^\circ < 90^\circ$ $\surd$	$\therefore x = 409^\circ > 90^\circ$ $\times$

$\therefore$ **The values of x are: 15°, 49° & 87°**

Example ❹

If $c \in \left]0^\circ , \frac{\pi}{2}\right[$ and if tan (2c + 15°) = cot (3c – 5°) , find the values of c

Solution

$\because \tan(2c + 15^\circ) = \cot(3c - 5^\circ)$

$\therefore (2c + 15^\circ) + (3c - 5^\circ) = 90^\circ + 180^\circ n$

$\therefore 2c + 15^\circ + 3c - 5^\circ = 90^\circ + 180^\circ n$

$\therefore 5c + 10^\circ = 90^\circ + 180^\circ n$

At n = 0

$\therefore 5c + 10^\circ = 90^\circ$ $\quad \therefore 5c = 80^\circ$ $\quad \therefore c = 16^\circ < 90^\circ$ $\quad \surd$

At n = 1

$\therefore 5c + 10^\circ = 90^\circ + 180^\circ$ $\quad \therefore 5c = 260^\circ$ $\quad \therefore c = 52^\circ < 90^\circ$ $\quad \surd$

At n = 2

$\therefore 5c + 10^\circ = 90^\circ + 360^\circ$ $\quad \therefore 5c = 440^\circ$ $\quad \therefore c = 88^\circ < 90^\circ$ $\quad \surd$

∴ The values of c are:16°, 52° & 88°

Example ❺

If $\cos\theta = \frac{1}{2}$ where $\theta \in \left]0^\circ , 360^\circ\right[$, find the possible values of θ

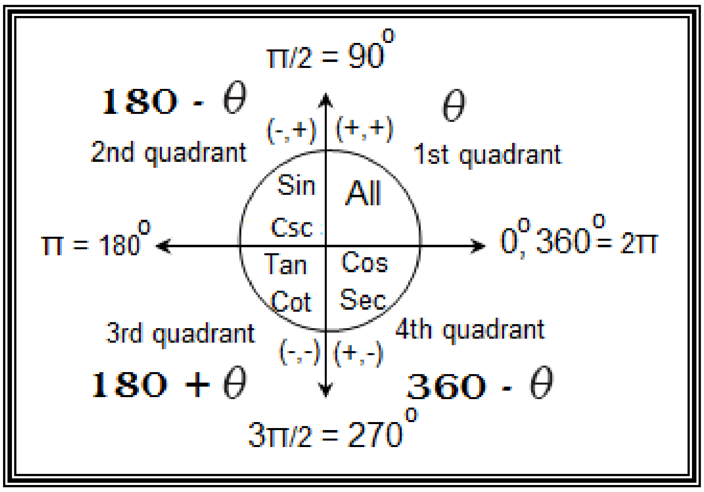
Solution

$\because \cos\theta = \frac{1}{2}$

∴ θ lies in the 1st or 4th quadrant

Let $\cos x = \frac{1}{2}$

$\therefore x = 60^\circ$



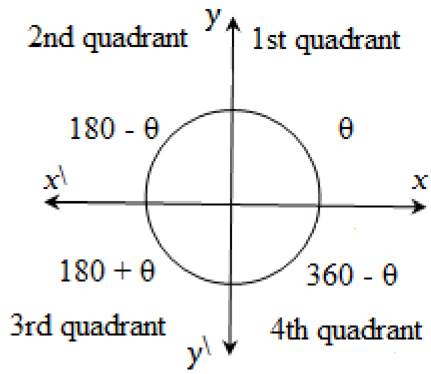
θ in 1st	θ in 4th
$\therefore \theta = x$	$\therefore \theta = (360^\circ - x)$
$\therefore \theta = 60^\circ$	$\therefore \theta = 300^\circ$

∴ The values of θ are:60° & 300°

Example ❻

Find the S.S. of the equation: $4\cos^2 x - 3 = 0$, where $x \in [\,0\,,\,2\,\pi\,]$

Solution



$\therefore \sin^2 x = \frac{3}{4}$

$\therefore \sin x = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$

$\therefore \sin x = \frac{\sqrt{3}}{2}$

$\therefore x$ lies in 1st or 2nd quad.

Let $\sin\theta = \frac{\sqrt{3}}{2}$ $\quad\therefore \theta = 60°$

$\therefore \sin x = -\frac{\sqrt{3}}{2}$

$\therefore x$ lies in 3rd or 4th quad.

Let $\sin\theta = \frac{\sqrt{3}}{2}$ $\quad\therefore \theta = 60°$

x in 1st	x in 2nd	x in 3d	x in 4th
$x = \theta$	$x = 180 - \theta$	$x = 180 + \theta$	$x = 360 - \theta$
$\therefore x = 60°$	$\therefore x = 120°$	$\therefore x = 240°$	$\therefore x = 300°$

$\therefore$ S.S. = { ° , 120° , 240° , 300° }

Example ❼

Find the general solution of the equation: $\sin 2\theta = \cos 5\theta$, which satisfies the equation.

Solution

$\because \sin 2\theta = \cos 5\theta$

$\therefore 5\theta \pm 2\theta = \frac{\pi}{2} + 2\pi\, n$

$\therefore 5\theta \pm 2\theta = 90° + 360°\, n$

$\therefore 5\theta + 2\theta = \frac{\pi}{2} + 2\pi\, n$

$\therefore 7\theta = \frac{\pi}{2} + 2\pi\, n$

$\therefore \theta = \frac{\pi}{14} + \frac{2\pi}{7}\, n$

$\therefore 5\theta - 2\theta = \frac{\pi}{2} + 2\pi\, n$

$\therefore 3\theta = \frac{\pi}{2} + 2\pi\, n$

$\therefore \theta = \frac{\pi}{6} + \frac{2\pi}{3}\, n$

PRACTICE (1)

Q1: Simplify $\cos(360^\circ - \theta)$.

A $-\cos\theta$
B $\cos\theta$
C $\sin\theta$
D $-\sin\theta$

Q2: Simplify $\tan(180^\circ - \theta)$.

A $-\tan\theta$
B $\tan\theta$
C $\cot\theta$
D $-\cot\theta$

Q3: Using the fact that $\cos\theta = \sin(90^\circ - \theta)$, which of the following is equivalent to $\cos 35^\circ$?

A $\dfrac{1}{\sin 35^\circ}$
B $-\sin 35^\circ$
C $\sin 35^\circ$
D $\sin 145^\circ$
E $\sin 55^\circ$

Q4: Simplify $\tan(360^\circ - \theta)$.

A $\tan\theta$
B $-\tan\theta$
C $\cot\theta$
D $-\cot\theta$

Q5: Which of the following is equivalent to $\sin 23°$?

A $-\cos 23°$

B $\cos 157°$

C $\cos 23°$

D $\dfrac{1}{\cos 23°}$

E $\cos 67°$

Q6: Which of the following is equivalent to $\cos 40°$?

A $\sin 140°$

B $-\sin 40°$

C $\sin 50°$

D $\sin 40°$

E $\dfrac{1}{\sin 40°}$

Q7: Which of the following is equal to $-\cos\theta$?

A $\sin\left(\frac{3\pi}{2}+\theta\right)$

B $\sin\left(\frac{\pi}{2}+\theta\right)$

C $\cos\left(\frac{3\pi}{2}+\theta\right)$

D $\cos\left(\frac{\pi}{2}+\theta\right)$

Q8: Which of the following is equal to $-\sin\theta$?

A $\cos\left(\frac{\pi}{2}+\theta\right)$

B $\sin\left(\frac{3\pi}{2}+\theta\right)$

C $\sin\left(\frac{\pi}{2}+\theta\right)$

D $\cos\left(\frac{3\pi}{2}+\theta\right)$

PRACTICE (2)

Q1: Find $\sin\theta$ given $51\cos(90^\circ-\theta)=24$ where θ is a positive acute angle.

A $-\frac{8}{17}$

B $-\frac{15}{17}$

C $\frac{15}{17}$

D $\frac{8}{17}$

Q2: Which of the following is equal to $\sin\theta$?

A $\cos\left(\frac{\pi}{2}+\theta\right)$

B $\cos\left(\frac{3\pi}{2}+\theta\right)$

C $\sin\left(\frac{3\pi}{2}+\theta\right)$

D $\sin\left(\frac{\pi}{2}+\theta\right)$

Q3: Find the value of $\tan(270^\circ-\theta)$ given $\cos\theta=-\frac{4}{5}$ where $90^\circ<\theta<180^\circ$.

A $\frac{4}{3}$

B $-\frac{4}{3}$

C $-\frac{3}{4}$

D $\frac{3}{4}$

Q4: Find the value of $\cos(90^\circ+\theta)$ given $\sin\theta=\frac{3}{5}$ where $0^\circ<\theta<90^\circ$.

A $-\frac{4}{5}$

B $\frac{4}{5}$

C $\frac{3}{5}$

D $-\frac{3}{5}$

Q5: Find the value of $\sin(180° - x) + \tan(360° - x) + 7\sin(270° - x)$ given $\sin x = \frac{3}{5}$ where $0° < \theta < 90°$

.

A $\frac{23}{4}$

B $\frac{139}{20}$

C $-\frac{139}{20}$

D $-\frac{23}{4}$

Q6: Find the value of $\sin(-60°)\cos 30° + \frac{\tan 57°}{\cot 33°}$ giving the answer in its simplest form.

A $-\frac{1}{4}$

B $\frac{3}{4}$

C $\frac{1}{4}$

D $-\frac{3}{4}$

Q7: Find the value of $\frac{\sin(90 - x)\sin(x)}{\cos(90 - 2x)}$.

A 1

B $\cos(90 - x)$

C $\frac{1}{2}$

D 2

E $\sin(2x)$

PRACTICE (3)

Q1: Find the value of $\cos 135° + \tan 135° + \csc 225° + \cos 225°$.

A $1+\sqrt{2}$

B $-\sqrt{2}-1$

C $1+2\sqrt{2}$

D $-2\sqrt{2}-1$

Q2: Calculate $\sin 315° \cos 45° - \cos 120° \sin 330°$.

A $\frac{3}{4}$

B $\frac{1}{4}$

C $-\frac{1}{4}$

D $-\frac{3}{4}$

Q3: Calculate $4 \sin 330° \sin^2 240° - \cos 270° \sec 240° + \sin 270° \cos^2 135°$.

A 1

B -1

C -2

D 2

Q4: Evaluate $\sin 150° \cos(-240)° + \cos 2\,130° \cot 240°$.

A $\frac{1}{4}$

B $-\frac{1}{4}$

C $-\frac{3}{4}$

D $\frac{3}{4}$

Related angles

4 - 4

First: Complete each of the following:

1. $\cos(180^\circ + \theta) =$ ....
2. $\tan(180^\circ - \theta) =$ ....
3. $\csc(360^\circ - \theta) =$ ....
4. $\sin(360^\circ + \theta) =$ ....
5. $\sin(90^\circ + \theta) =$ ....
6. $\cot(90^\circ - \theta) =$ ....
7. $\sec(270^\circ + \theta) =$ ....
8. $\cos(270^\circ - \theta) =$ ....

Second: Complete each of the following with a measure of an acute angle

9. $\sin 25^\circ = \cos$ ....$^\circ$
10. $\cos 67^\circ = \sin$ ....$^\circ$
11. $\tan 42^\circ = \cot$ ....$^\circ$
12. $\csc 13^\circ = \sec$ ....$^\circ$
13. If $\cot an\ 2\theta = \tan\theta$ where $0^\circ < \theta < 90^\circ$ then $m(\angle\ \theta) =$ ....
14. If $\sin 5\theta = \cos 4\theta$ where θ is a positive acute angle , then $\theta =$ ....$^\circ$
15. If $\sec\theta = \sec(90^\circ - \theta)$, then $\cot\theta =$ ....
16. If $\tan 2\theta = \cot 3\theta$ where $\theta \in\ ]0, \frac{\pi}{2}[$, then $m(\angle\ \theta) =$ .... rad
17. If $\cos\theta = \sin 2\theta$ where θ is a positive acute angle, then $\sin 3\theta =$ ....

Third: Multiple choice:

(18) If $\tan(180^\circ + \theta) = 1$ where θ is the measure of the smallest positive angle , then measure of θ equals

A 45° B 30° C 60° D 135°

(19) If $\cos 2\theta = \sin\theta$ where $\theta \in]0, \frac{\pi}{2}[$ then $\cos 2\theta$ equals ....

A $\frac{1}{\sqrt{2}}$ B $\frac{1}{2}$ C $\frac{\sqrt{3}}{2}$ D 1

(20) If $\sin\alpha = \cos\beta$ where α and β are two acute angles, then $\tan(\alpha + \beta)$ equals ....

A $\frac{1}{\sqrt{3}}$ B 1 C $\sqrt{3}$ D undefined

(21) If $\sin 2\theta = \cos 4\theta$ where θ is a positive acute angle, then $\tan(90^\circ - 3\theta)$ equals..........

A -1 B $\frac{1}{\sqrt{3}}$ C 1 D $\sqrt{3}$

(22) If $\cos(90^\circ + \theta) = \frac{1}{2}$ where θ is the measure of the smallest positive angle , then measure of angle θ equals..........

A 150° B 210° C 240° D 330°

Fourth: Answer the following question:

(23) Find one of the values of θ where $0 \leqslant \theta < 90^\circ$ which satisfies each of the following:

A $\sin(3\theta + 15^\circ) = \cos(2\theta - 5^\circ)$ ....

B $\sec(\theta + 25^\circ) = \csc(\theta + 15^\circ)$ ....

C $\tan(\theta + 20^\circ) = \cot(3\theta + 30^\circ)$ ....

D $\cos\frac{\theta + 20^\circ}{2} = \sin\frac{\theta + 40^\circ}{2}$ ....

(24) Find the value of each of the following:

A $\sin 150^\circ$ .... B $\csc 225^\circ$ .... C $\sec 300^\circ$ .... D $\tan 780^\circ$ ....

E $\csc\frac{11\pi}{6}$ .... F $\sin\frac{7\pi}{4}$ .... H $\cot\frac{-2\pi}{3}$ .... I $\cos\frac{-7\pi}{4}$ ....

25 If the terminal side of the angle θ drawn in the standard position intersects the unit circle at the point B $(-\frac{3}{5}, \frac{4}{5})$, then find:

A $\sin(180^\circ + \theta)$ ....

B $\cos(\frac{\pi}{2} - \theta)$ ....

C $\tan(360^\circ - \theta)$ ....

D $\csc(\frac{3\pi}{2} - \theta)$ ....

26 **Discover the error:** All the following answers are correct except one wrong. What is it?:

1- $\cos\theta$ equals ....

A $\sin(\theta - 270^\circ)$ **B** $\sin(270^\circ - \theta)$ **C** $\cos(360^\circ - \theta)$ **D** $\cos(360^\circ + \theta)$

2- $\sin\theta$ equals ....

A $\cos(\frac{\pi}{2} - \theta)$ **B** $\sin(\pi - \theta)$ **C** $\cos(\frac{3\pi}{2} + \theta)$ **D** $\sin(\frac{\pi}{2} + \theta)$

3- $\tan\theta$ equals ....

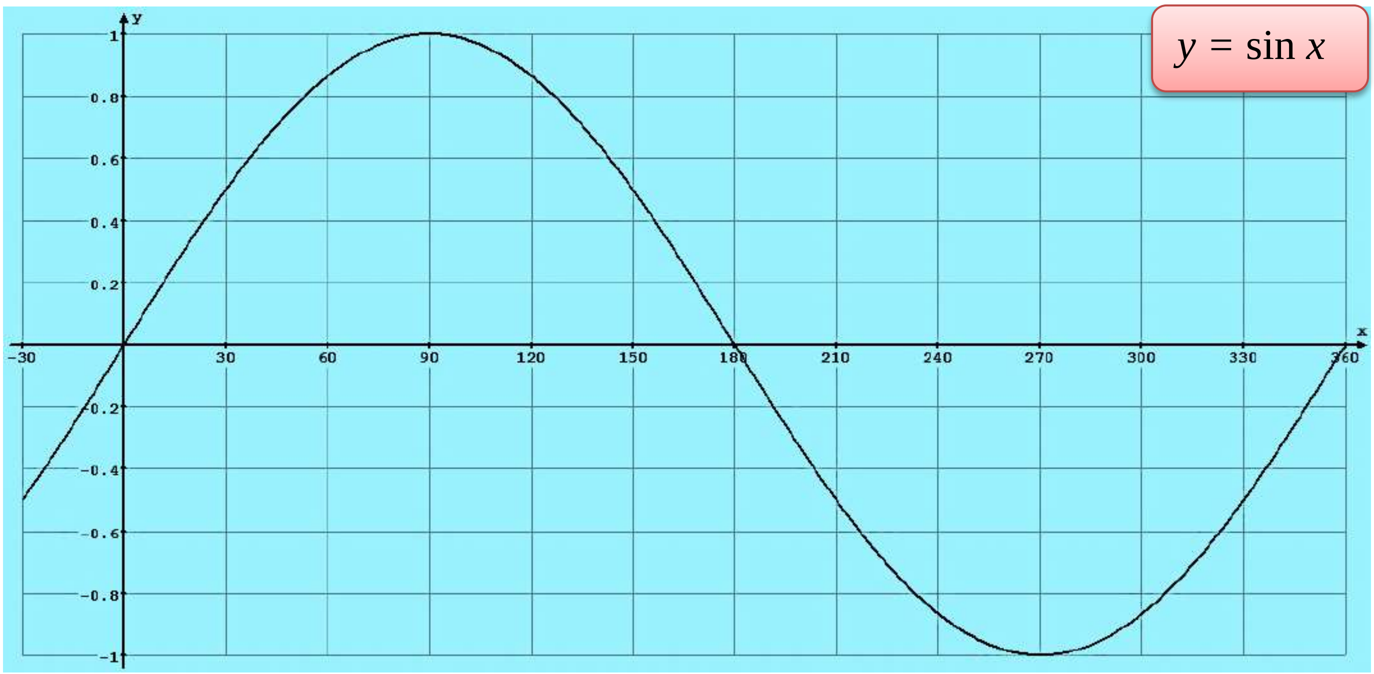
A $\cot(90^\circ - \theta)$ **B** $\cot(270^\circ - \theta)$ **C** $\tan(270^\circ - \theta)$ **D** $\tan(180^\circ + \theta)$

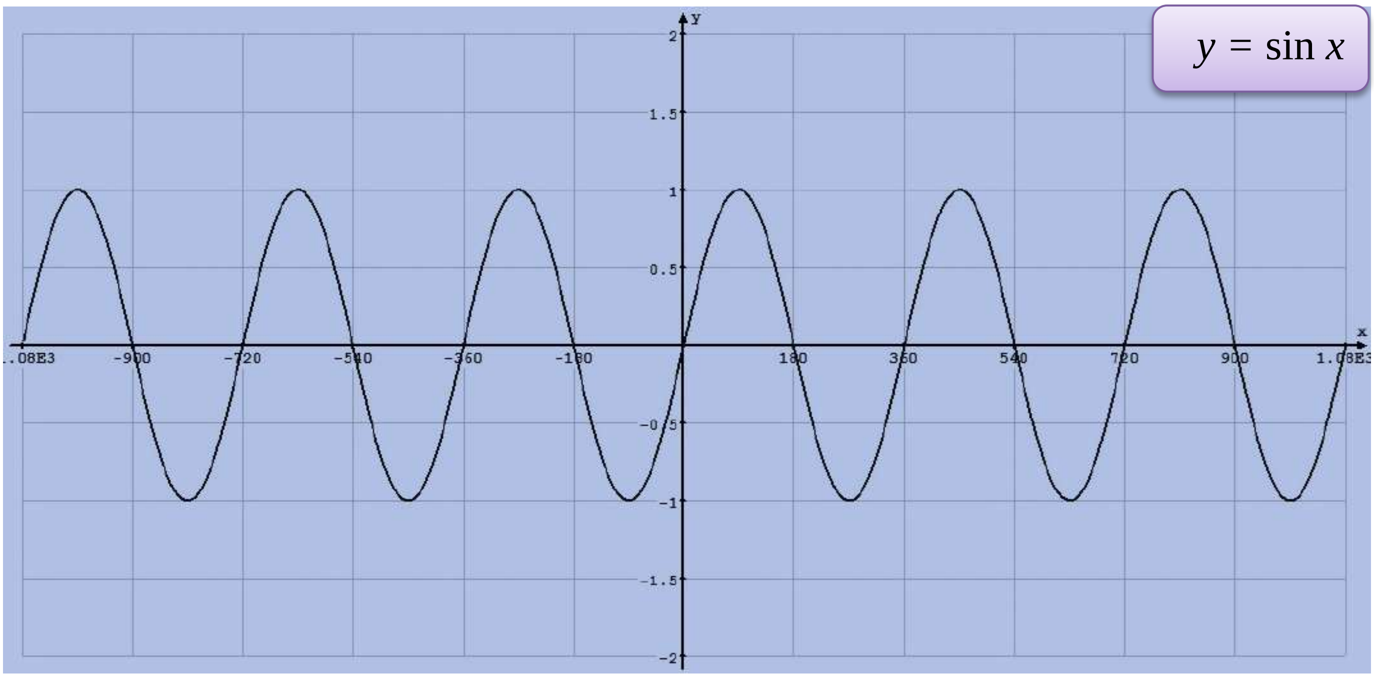
Lesson (5) Graphing trigonometric functions

Lesson objectives

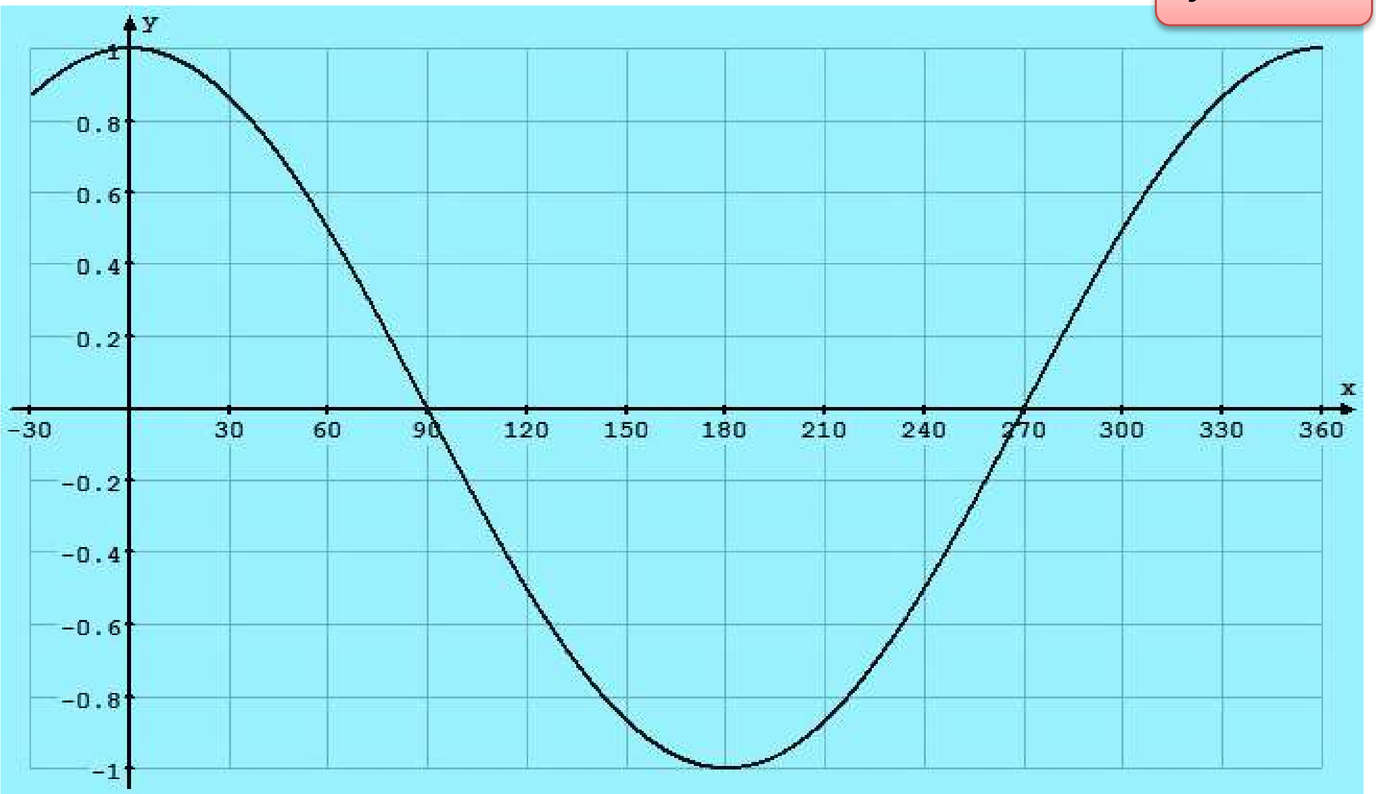
Related Links

- Graph the sine function , and deduce its properties.
- Graph the cosine function, and deduce its properties.

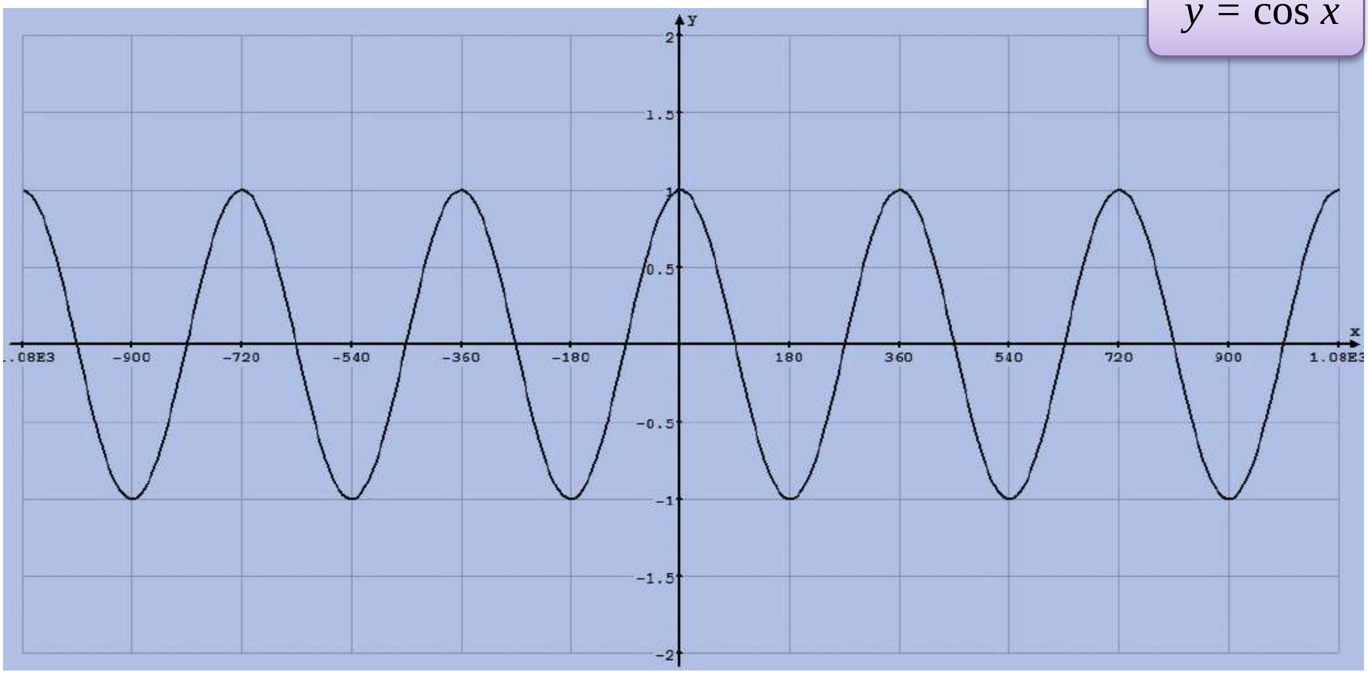




$y = \cos x$


Y
1
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-1
x
-30
30
60
90
120
150
180
210
240
270
300
330
360

$y = \cos x$


Y
2
1.5
1
0.5
-0.5
-1
-1.5
-2
x
-900
-720
-540
-360
-180
180
360
540
720
900

Example ❶

Graph the function $y = 5 \sin 3x$, where $0^\circ \leq x \leq 120^\circ$

Solution

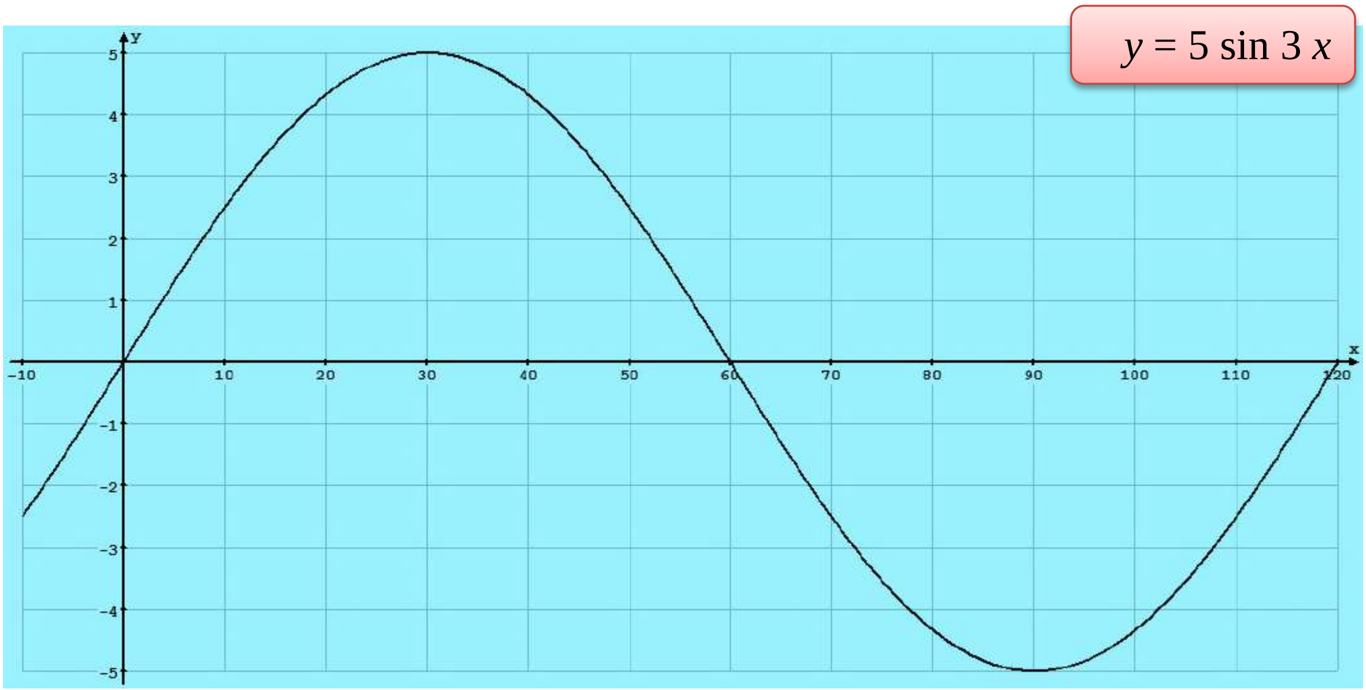
$\because 0^\circ \leq x \leq 120^\circ$ $\qquad \therefore 0^\circ \leq 3x \leq 360^\circ$

① We start to fill second row with angles: 0° , 30° , 60° , 90° , 120° , …. , 360°

② We fill first row with angles : 0° , 10° , 20° , 30° , 40° , …… , 120°

③ We fill in the third row by the values of the sine of the angles in 2nd row:

x	0	10	20	30	40	50	60	70	80	90	100	110	120
$3x$	0	30	60	90	120	150	180	210	240	270	300	330	360
$\sin 3x$	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0
$5 \sin 3x$	0	2.5	4.35	5	4.35	2.5	0	-2.5	-4.35	-5	-4.35	-2.5	0



Remark In the form of:

a sin b x & a cos b x , then the range is: [-a , a] & Its periodic = $\frac{360°}{b} = \frac{2\pi}{b}$

Example ❷

Complete each of the following:

(1) If $f(x) = \cos 5x$, then the range of the function is ………………… $[-1, 1]$

(2) If $f(x) = 4 \sin x$, then the range of the function is ………………… $[-4, 4]$

(3) The function $f: f(x) = \cos 2x$ is a periodic function and its period is ….

Periodic = $\frac{360°}{2} = 180°$

(4) The function $f: f(x) = 5 \cos 3x$ is a periodic function and its period is….

Periodic = $\frac{360°}{3} = 120°$

(5) If $x \in [0, \pi]$, then: ………… $\leq \sin x \leq$ ………… $0 \leq \sin x \leq 1$

(6) If $x \in [0, 2\pi]$, then: ……… $\leq \cos x \leq$ …………… $-1 \leq \cos x \leq 1$

(7) If $\cos 2x \in [-1, 1]$, then $\cos 4x \in$ […….. , ……..] [-1, 1]

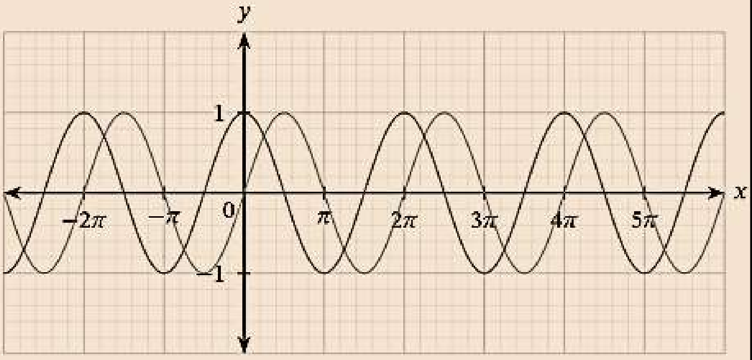
(8) If the function $y = a \sin bx$ where a and b $\in \mathbb{R}_+$ is periodic and its period is 720° and its range $[-3, 3]$, then a = ………………… , b = …………………

$\because$ Its range = $[-3, 3]$ $\therefore$ a = 3

$\because$ Its period = 720° $\therefore \frac{360°}{b} = 720°$ $\therefore b = \frac{360°}{720°} = \frac{1}{2}$

PRACTICE

Q1: Assign each plot shown in the graph below to the function it represents.



A the red plot: cosine, the blue plot: sine

B the red plot: sine, the blue plot: cosine

C the red plot: cosine, the blue plot: tangent

D the red plot: tangent, the blue plot: cosine

E the red plot: tangent, the blue plot: sine

Q2: Find the range and period of the function $f(\theta) = \frac{14}{5}\sin 4\theta$ by drawing the function between 0 and 2π.

A range= $\left[0, \frac{14}{5}\right]$, period= $\frac{\pi}{2}$

B range= $[-1, 1]$, period= $\frac{\pi}{2}$

C range= $\left[-\frac{14}{5}, \frac{14}{5}\right]$, period= 2π

D range= $\left[-\frac{14}{5}, \frac{14}{5}\right]$, period= $\frac{\pi}{2}$

Q3: Find the range of the function $f(\theta) = 8\sin 7\theta$.

A $[-7, 7]$

B $[0, 1]$

C $[0, 8]$

D $[-8, 8]$

E $[-1, 1]$

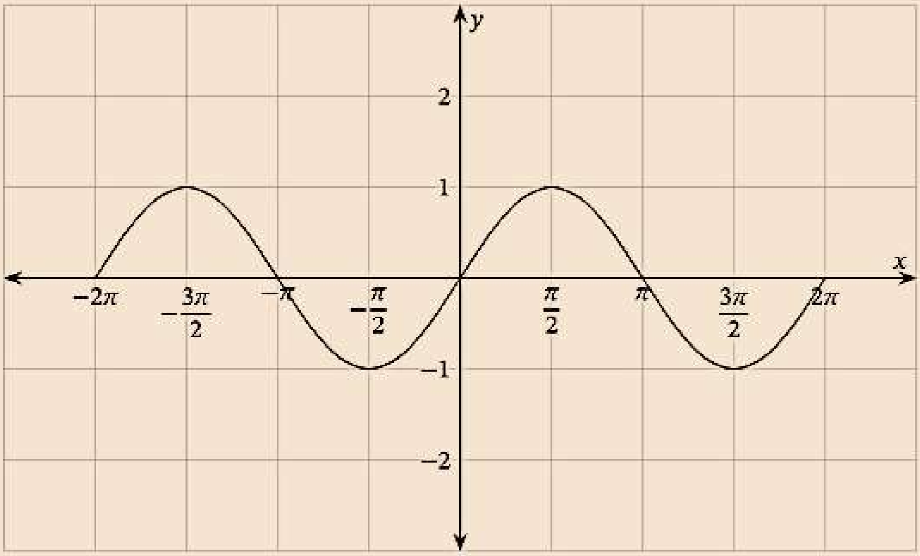
Q4: Find the maximum value of the function $f(\theta) = 11\sin\theta$.

Q5: Which function is represented on the graph?

A $y = 2\sin x$

B $y = \sin x$

C $y = \cos x$

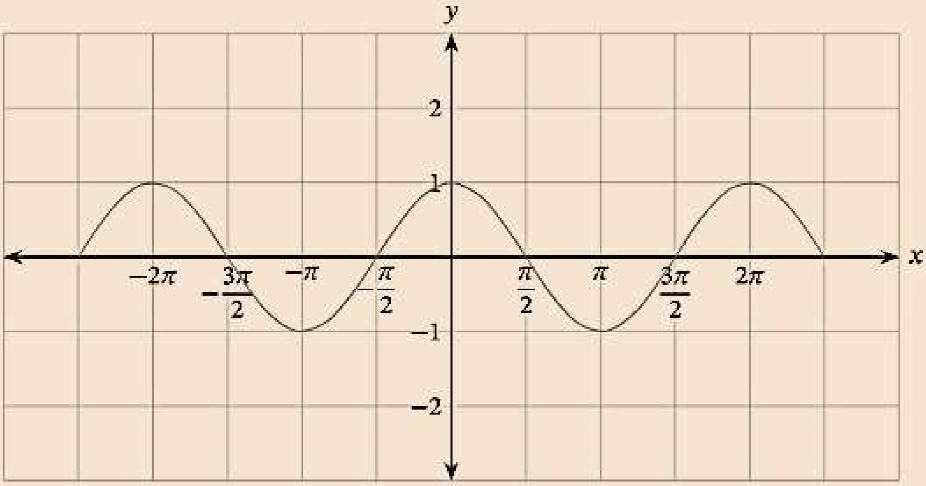


Q6: Which function is represented on the graph?

A $y = \cos x$

B $y = 2\sin x$

C $y = \sin x$



Q7: The range of the function $f(\theta) = a\cos 3\theta$ is $\left[-\frac{5}{4}, \frac{5}{4}\right]$. Find the value of a where $a > 0$.

A $-\frac{5}{4}$

B -3

C $\frac{5}{4}$

D 3

4 - 5 Graphing trigonometric functions

First: complete each of the following:

1. The range of the function f where $f(\theta) = \sin\theta$ is ....
2. The range of the function f where $f(\theta) = 2\sin\theta$ is ....
3. The maximum value of the function f where $f(\theta) = 4\sin\theta$ is ....
4. The minimum value of the function f where $f(\theta) = 3\cos\theta$ is ....

Second: write the rule for each trigonometric function beside the corresponding figure to it.

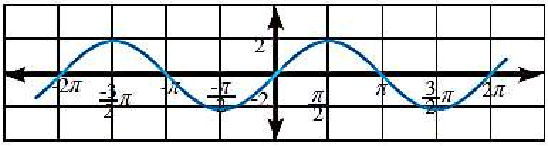


Figure (1) the rule is:

....

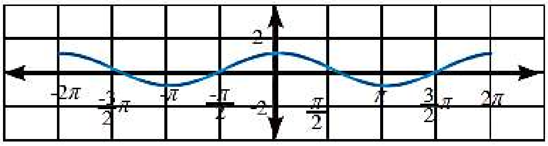


Figure (2) the rule is:

....

Third: Answer the following questions:

5. Find the maximum and minimum values, then calculate the range of each o the following functions :

A) $y = \sin\theta$

....

B) $y = 3\cos\theta$

....

C) $y = \frac{3}{2}\sin\theta$

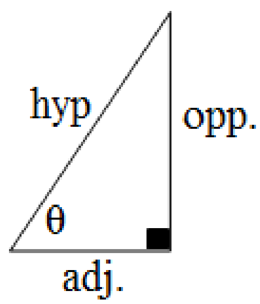
....

Lesson (6)

Trigonometric functions of an acute angle

REMEMBER THAT:

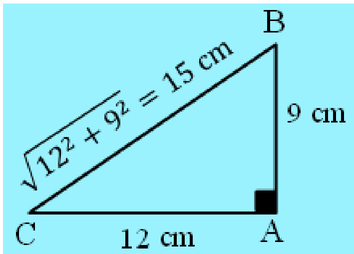
- $\sin\theta = \dfrac{opp}{hyp}$
- $\cos\theta = \dfrac{adj}{hyp}$
- $\tan\theta = \dfrac{opp}{adj}$
- $\csc\theta = \dfrac{hyp}{opp}$
- $\sec\theta = \dfrac{hyp}{adj}$
- $\cot\theta = \dfrac{adj}{opp}$



Example ❶

Δ ABC is a right-angled triangle at A where AB = 9 cm. , AC = 12 cm.
Find the measure of each of ∠ B and ∠ C

Solution



In Δ ABC:

$\because$ m (∠ A) = 90°

$\therefore BC = \sqrt{12^2 + 9^2} = 15$ cm

$\because \sin B = \dfrac{12}{15}$ $\quad\therefore m(\angle B) = \sin^{-1}\dfrac{12}{15} = 53°\ 7'\ 48''$

$\because \tan C = \dfrac{9}{12}$ $\quad\therefore m(\angle C) = \tan^{-1}\dfrac{12}{15} = 38°\ 39'\ 35''$

Example ❷

If 3 csc θ – 5 = 0 , where 0° < θ < 90° , find the value of : sec θ – tan θ

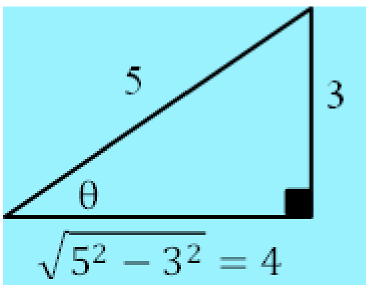
Solution

$\because 3\csc\theta - 5 = 0$

$\therefore \csc\theta = \dfrac{5}{3}$

Hypotenuse

Opposite



$\therefore \sec\theta - \tan\theta = \dfrac{5}{4} - \dfrac{3}{4} = \dfrac{1}{2}$

Example ❸

Δ ABC is a triangle in which:

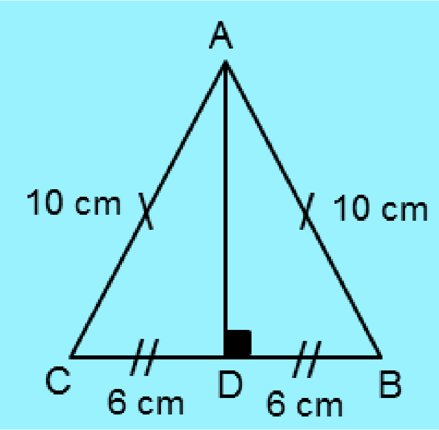
AB = AC = 10 cm. , AC = 12 cm. $\overrightarrow{AD}$ is drawn $\perp$ $\overline{BC}$ to cut it at D

❶ Find the value of: sin B + cos C

❷ Find the value of: tan ($\angle$ CAD)

❸ Show that: sin C + cos C > 1 , then find the value of: $\sin^2 C + \cos^2 C$ and deduce that: $\sin^2 C + \cos^2 < \sin C + \cos C$

Solution



$\because \overrightarrow{AD} \perp \overline{BC}$

$\therefore AD = \sqrt{10^2 - 6^2} = 8$ cm

❶ $\because \sin B = \frac{8}{10} = \frac{4}{5}$ & $\cos C = \frac{6}{10} = \frac{3}{5}$

$\therefore \sin B + \cos C = \frac{4}{5} + \frac{3}{5} = \frac{7}{5}$

❷ In Δ ACD:

$\therefore \tan(\angle CAD) = \frac{6}{8} = \frac{3}{4}$

❸ $\because \sin C = \frac{8}{10} = \frac{4}{5}$ & $\cos C = \frac{6}{10} = \frac{3}{5}$

$\therefore \sin C + \cos C = \frac{4}{5} + \frac{3}{5} = \frac{7}{5} = 1.6$

$\therefore \sin C + \cos C > 1$

$\therefore \sin^2 C + \cos^2 C = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = 1$

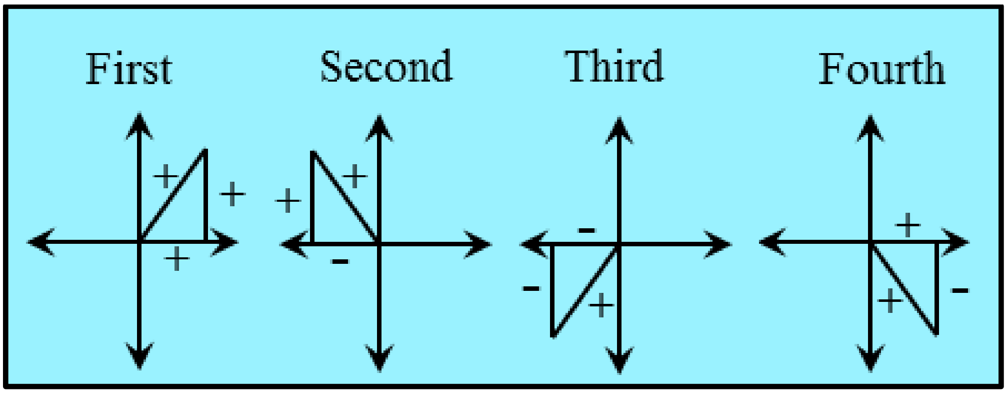
$\therefore \sin^2 C + \cos^2 < \sin C + \cos C$

Lesson (6)   Finding the measure of an angle

Lesson objectives

Related Links

- Find the measure of an angle given a trigonometric function.



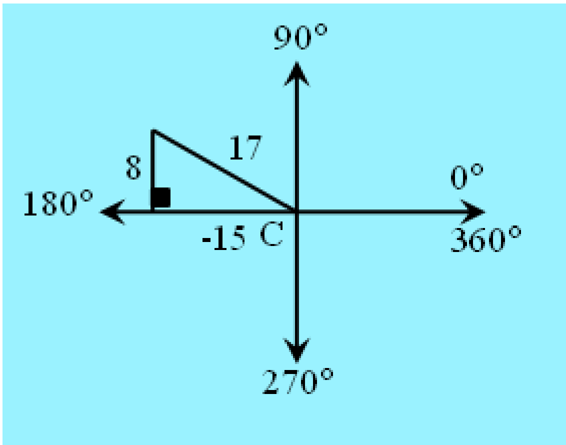
Example ❶

If: $\sin C = \frac{8}{17}$, where $90^\circ < C < 180^\circ$, find the other trigonometric functions of angle C

Solution

$\because \sin C = \frac{8}{17}$ — Opposite / Hypotenuse

$\therefore$ C lies on 2^{nd} quadrant



$\sin C = \frac{8}{17}$	$\cos C = -\frac{15}{17}$	$\tan C = -\frac{8}{15}$
$\csc C = \frac{17}{8}$	$\sec C = -\frac{17}{15}$	$\cot C = -\frac{15}{8}$

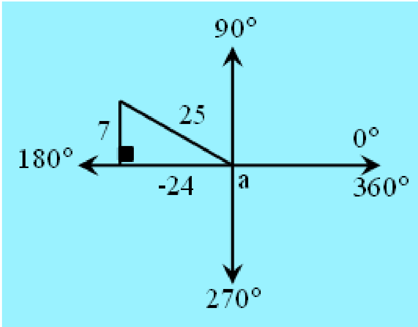
Example ❷

If $\angle$ a is the greatest positive angle where:

25 sin a = 7 and 5 tan b − 12 = 0 , where b ∈]180° , 270°[, Find m ($\angle$ c) if:

tan c = sin (180° – a) cos (90° – b) – $\frac{1}{6}$ cos (– a) sin (270° – b)

Solution

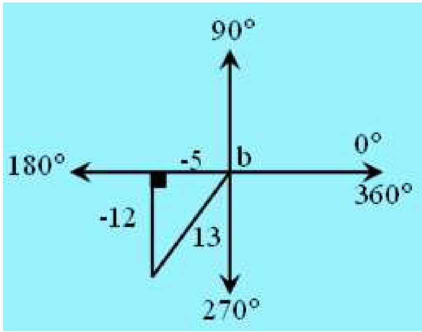


∵ 25 sin a = 7 **∴ sin a = $\frac{7}{25}$**

∴ a lies in 1st or 2nd quadrant.

∵ $\angle$ a is the greatest positive angle

∴ a lies in 2nd quadrant.



∵ 5 tan b − 12 = 0 **∴ tan b = $\frac{12}{5}$**

∴ b lies in 2nd or 4th quadrant.

∵ b ∈]180° , 270°[

∴ b lies in 3rd quadrant.

∴ tan c = sin (180° – a) cos (90° – b) – $\frac{1}{6}$ cos (– a) sin (270° – b)

∴ tan c = sin a × sin b – $\frac{1}{6}$ × cos a × (− cos b)

∴ tan c = sin a × sin b + $\frac{1}{6}$ × cos a × cos b

$$= \frac{7}{25} \times \frac{-12}{13} + \frac{1}{6} \times \frac{-24}{25} \times \frac{-5}{13} = \frac{-64}{325}$$

∵ tan c = $\frac{-64}{325}$ **∴ c lies in 2nd or 4th quadrant.**

Let tan θ = $\frac{64}{325}$ **∴ $\theta = 11^\circ\ 8'\ 25''$**

∴ c in 2nd quadrant = 180° − θ = $168^\circ\ 51'\ 35''$

∴ c in 4th quadrant = 360° − θ = $348^\circ\ 51'\ 35''$

PRACTICE (1)

Q1: Find $\csc\theta$ given $\tan\theta = \frac{24}{7}$ and $\cos\theta < 0$.

A $-\frac{25}{24}$

B $-\frac{25}{7}$

C $\frac{25}{24}$

D $\frac{25}{7}$

Q2: Find the value of $\cos(180° - \theta)$ given $\sin\theta = -\frac{3}{5}$ where $270° < \theta < 360°$.

A $\frac{4}{5}$

B $-\frac{4}{5}$

C $\frac{3}{4}$

D $-\frac{3}{4}$

Q3: Find all the trigonometric ratios of θ given $\cot\theta = -\frac{8}{15}$ where $\theta \in \left]\frac{3\pi}{2}, 2\pi\right[$.

A $\sin\theta = -\frac{15}{17}, \cos\theta = \frac{8}{17}, \tan\theta = \frac{15}{8}, \csc\theta = \frac{17}{15}, \sec\theta = \frac{17}{8}$

B $\sin\theta = \frac{15}{17}, \cos\theta = \frac{8}{17}, \tan\theta = -\frac{15}{8}, \csc\theta = -\frac{17}{15}, \sec\theta = -\frac{17}{8}$

C $\sin\theta = -\frac{15}{17}, \cos\theta = \frac{8}{17}, \tan\theta = -\frac{15}{8}, \csc\theta = -\frac{17}{15}, \sec\theta = \frac{17}{8}$

D $\sin\theta = -\frac{15}{17}, \cos\theta = -\frac{8}{17}, \tan\theta = \frac{15}{8}, \csc\theta = -\frac{17}{15}, \sec\theta = \frac{17}{8}$

Q4: Given that $\cot(\theta) = -\frac{3}{2}$, where $\frac{\pi}{2} < \theta < \pi$, evaluate $\sec^2(\theta)$ without using a calculator.

A $\frac{13}{9}$

B $\frac{9}{13}$

C $-\frac{9}{13}$

D $-\frac{13}{9}$

E $-\frac{5}{2}$

Q5: Given that $\csc\theta = -\frac{7}{6}$ and $\tan\theta > 0$, find $\cos\theta$.

A $\frac{\sqrt{13}}{7}$

B $-\frac{6}{7}$

C $-\frac{\sqrt{13}}{6}$

D $\frac{6}{7}$

E $-\frac{\sqrt{13}}{7}$

PRACTICE (2)

Q1: Find the value of θ that satisfies $\csc\theta - \sqrt{2} = 0$ where $\theta \in \left]0, \frac{\pi}{2}\right[$.

Q2: Find the set of values satisfying $\sqrt{3}\cot\theta = 1$ given $0^\circ < \theta < 360^\circ$.

A $\{300^\circ, 240^\circ\}$

B $\{120^\circ, 300^\circ\}$

C $\{60^\circ, 300^\circ\}$

D $\{60^\circ, 240^\circ\}$

Q3: Find θ in degrees given $\sec(180^\circ + \theta) = -\frac{2\sqrt{3}}{3}$ where θ is the smallest positive angle.

Q4: Find the set of values satisfying $\sin\theta \cot\theta = -\frac{1}{2}$ where $0^\circ \leq \theta \leq 90^\circ$.

A $\{30^\circ, 330^\circ\}$

B $\varnothing$

C $\{30^\circ, 180^\circ\}$

D $\{180^\circ, 135^\circ\}$

Q5: Find the value of θ that satisfies $\sec\theta - 2 = 0$ where $\theta \in \left]0, \frac{\pi}{2}\right[$.

Q6: Find the value of θ that satisfies $\sec\theta - \frac{2\sqrt{3}}{3} = 0$ where $\theta \in \left]0, \frac{\pi}{2}\right[$.

Q7: Find the value of θ that satisfies $\csc\theta - \frac{2\sqrt{3}}{3} = 0$ where $\theta \in \left]0, \frac{\pi}{2}\right[$.

Q8: Find the set of values satisfying $\cot\theta = -1$ given $0^\circ < \theta < 360^\circ$.

A $\{135^\circ, 225^\circ\}$

B $\{225^\circ, 315^\circ\}$

C $\{45^\circ, 225^\circ\}$

D $\{135^\circ, 315^\circ\}$

Q9: Find θ in degrees given $\sin(180^\circ + \theta) = -\frac{\sqrt{3}}{2}$ where θ is the smallest positive angle.

Finding the measure of an angle given the value of one of its trigonometric ratios

4 - 6

First : Multiple choice:

1. If $\sin\theta = 0.4325$ where θ is a positive acute angle, then $m(\angle\theta)$ equals ....
   - (A) 25.626°
   - (B) 64.347°
   - (C) 32.388°
   - (D) 46.316°

2. If $\tan\theta = 1.8$ and $90^\circ \leqslant \theta \leqslant 360^\circ$, then $m(\angle\theta)$ equals ....
   - (A) 60.945°
   - (B) 119.055°
   - (C) 240.945°
   - (D) 299.055°

Second : Answer the following questions:

1. If the terminal side of angle θ in the standard position intersects the unit circle at point B, then find each of $\sin\theta$ and $\cos\theta$ in the following cases:
   - (A) $B\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
   - (B) $B\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
   - (C) $B\left(-\frac{6}{10}, \frac{8}{10}\right)$

2. If the terminal side of angle θ in the standard position intersects the unit circle at point B then find each of $\sec\theta$ and $\csc\theta$ in the following cases:
   - (A) $B\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
   - (B) $B\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$
   - (C) $B\left(-\frac{5}{13}, -\frac{12}{13}\right)$

3. If the terminal side of angle θ in the standard position intersects the unit circle at point B, then find each of $\tan\theta$ and $\cot\theta$ in the following cases::
   - (A) $B\left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$
   - (B) $B\left(\frac{3}{\sqrt{34}}, -\frac{5}{\sqrt{34}}\right)$
   - (C) $B\left(-\frac{4}{5}, -\frac{3}{5}\right)$

4 If the terminal side of angle θ in the standard position intersects the unit circle at point B, then find $m(\angle\theta)$ where $0^\circ < \theta < 360^\circ$ when:

A $B\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ .... **B** $B\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ .... **C** $B\left(\frac{6}{10}, \frac{-8}{10}\right)$ ....

5 Use the degree measure to find the smallest positive angle which satisfies each of the following:

A $\sin^{-1} 0.6$ .... **B** $\cos^{-1} 0.436$ .... **C** $\tan^{-1} 1.4552$ ....

D $\sec^{-1} (-2.2364)$ .... **E** $\cot^{-1} 3.6218$ .... **F** $\csc^{-1} (-1.6004)$ ....

6 If $0^\circ \leqslant \theta \leqslant 360^\circ$, then find the measure of angle θ in each of the following:

A $\sin^{-1} (0.2356)$ .... **B** $\cos^{-1} (-0.642)$ .... **C** $\tan^{-1} (-2.1456)$ ....

7 If $\sin\theta = \frac{1}{3}$ and $90^\circ \leqslant \theta \leqslant 180^\circ$.

A Calculate the measure of angle θ to the nearest second ....

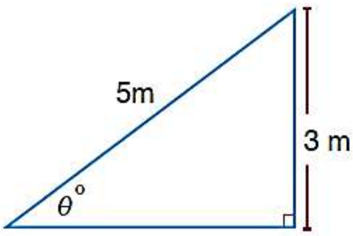
B Find the value of $\cos\theta$, $\tan\theta$ and $\sec\theta$.

....

....

....

8 **Ladder:** A ladder of length 5 metres rests on a wall , if the height of the ladder from the ground is 3 metres. Find in radian the measure of the angle of inclination of the ladder to the horizontal.



....

Unit Test

Answer the following question and approximate the result to the nearest hundredth:

1 Convert the following angles from degree to radian measure:

A 120° .... **B** 64.8° .... **C** $220^\circ\ 36'$ ....

2 Convert the following angles from radian to degree measure:

A $\frac{5\pi}{3}$ .... **B** $-\frac{3\pi}{2}$ .... **C** 1.12^{rad} ....

3 θ is a central angle in a circle of radius length r and subtends an arc length L:

A If $r = 8$ cm and $\theta = 1.2^{rad}$, then find L. ....

B If $L = 26$ cm and $r = 18$ cm, then find θ in degree measure. ....

4 Without using the calculator , find the value of each of the following:

A $\tan 120^\circ$ **B** $\sin\left(\frac{13\pi}{6}\right)$ **C** $\cos 330^\circ$ **D** $\cot(-300^\circ)$ **E** $\csc\left(-\frac{\pi}{3}\right)$

....

5 Find all the trigonometric function of angle θ drawn in the standard position and its terminal side intersects the unit circle in each of the following points:

A $\left(\frac{4}{5}, \frac{3}{5}\right)$ **B** $\left(\frac{-5}{13}, \frac{-12}{13}\right)$ **C** $\left(\frac{-3}{5}, \frac{-4}{5}\right)$ **D** $\left(\frac{-\sqrt{5}}{3}, \frac{2}{3}\right)$

....

6 **A** Prove that:

First: $\sin 60 = 2 \sin 30^\circ \cos 30^\circ$. **second:** $\cos 300^\circ = 2 \sin^2 60^\circ - 1$

B If $\cos\theta = -\frac{4}{5}$ where $90^\circ < \theta < 180^\circ$, then find the value of each of the following:

First: $\sin(180^\circ - \theta)$ **Second:** $\tan(\theta - 180^\circ)$

....

7) Find the degree measure in the interval $0^\circ \leqslant \theta \leqslant 360^\circ$ for each of the following:

A) $\tan^{-1} 1$ ……………………

B) $\sin^{-1}(-\frac{1}{2})$ ……………………

C) $\cos^{-1}(\frac{\sqrt{3}}{2})$ ……………………

D) $\tan^{-1}(-\sqrt{3})$ ……………………

8) A ramp length is 24 metres and its height from the ground is 9 metres. Write a trigonometric function you can use to find the measure of the angle of inclination of the ramp on the horizontal ground, then find its measure.

……………………………………………………………………………………………………